Sets, Relations and Probability. Part IA Formal Methods.

Lecture I, *Introduction to Basic Set Theory*

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In the following two lectures, we'll discuss the core elements of basic set theory. Today, we'll discuss what a set is, some notation used in set theory, and some basic laws governing sets.

1. Introduction: What is a set?

1.1. It's useful to start with a metaphor. Think of a set as a big box in which you can put anything you like. Just as boxes contain things, so do sets. If a set contains an object, we say that the set has the object as a *member*. This metaphor is useful in emphasising some features of sets:

– A set is a certain *kind of collection*.

– A set is a collection of things *considered as a single object*

So, just like a box of marbles which contains *many* marbles inside *one single* box, the set containing the same marbles contains many marbles, and yet the set itself is a single thing. A set is not just the things in contains, nor is it a *fusion* of the things it contains.

1.2. The metaphor is limited, however. In particular, there are significantly fewer limits on what can be contained in a set compared with what can go in any (ordinary) box. For instance, there's the set of *all* cats, the set of *all* dogs, and the set of *all* cats *and* dogs. There are also less 'natural' sets. So, there's the set containing my left pinky finger, the Empire State Building, and nothing else. Indeed, what's in a set doesn't even have to be physical: there are sets of abstract objects like the set of all natural numbers or the set of all even numbers. Sets can also contain *other sets* and, as we'll see, sets may even contain *themselves*.

1.3. If some x is a member of a set y, we write $x \in y$. If the number of things contained in a set are finite, then we can write the set with the list of the things it contains enclosed in curly brackets. So:

- The set $\{a, b, c\}$ is the set containing a, b, and c and nothing else. **Example 1**
  - So, we write: $a \in \{a, b, c\}$ and $b \in \{a, b, c\}$ and $c \in \{a, b, c\}$.
  - Note that d is *not* in $\{a, b, c\}$. We write this as $\neg d \in \{a, b, c\}$ or $d \notin \{a, b, c\}$.

Sets can be members of other sets. So, again, we can also sometimes write this like above:

- The set $\{a, b, \{c\}\}$ is the set containing a, b, and $\{c\}$ and nothing else. **Example 2**
  - So, we write: $a \in \{a, b, \{c\}\}$ and $b \in \{a, b, \{c\}\}$ and $\{c\} \in \{a, b, \{c\}\}$.
  - Note that c is not in $\{a, b, \{c\}\}$. c and $\{c\}$ are different. Only $\{c\} \in \{a, b, \{c\}\}$.

1.4. How do we *individuate* sets? That is, when are two sets identical to each other? The answer is a basic law governing sets, *The Axiom of Extensionality*—two sets are identical just in case they have precisely the same members. In other symbols, we express this idea as follows, where S is '...is a set':

AXIOM OF EXTENSIONALITY: $\forall x \forall y((Sx \wedge Sy) \rightarrow (x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y))$

(Read: For two sets x and y, x is y if and only if, for any z, z is in x if and only if z is in y.)

Thus, if two x and y are distinct, it follows that there is *something* which is not a member of both x and y. Extensionality ensures that sets are not the fusion of their members. The set of the quartered segments of an apple is distinct from the set of the halved segments of an apple; but the fusion in both cases is the apple.

Extensionality ensures that the order in which we write the members of a set doesn't matter: $\{a, c\} = \{c, a\}$.

- Are the sets $s_1 = \{a, b, c\}$ and the set $s_2 = \{a, b, c, c\}$ distinct? **Question 1**
  - ⋆ No, the two sets are identical. This follows from extensionality. Any member of $\{a, b, c\}$ is a member of $\{a, b, c, c\}$ and *vice versa*. That is, each of a, b, and c is a member of s_1 if and only if it is a member of s_2. Therefore, by extensionality they are identical.

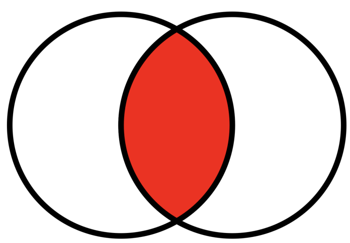
Note: We *do not* write repetition of the members of a set like $\{a, b, c, c\}$. We just write $\{a, b, c\}$.

2. Some Important Further Principles Governing Sets

2.1. For any two sets x and y, there is a set called the *intersection*. The intersection of x and y is the set containing the things which are members of *both* x and y. We write the intersection as $x \cap y$.

INTERSECTION: $\forall x \forall y \forall z(z \in x \cap y \leftrightarrow (z \in x \wedge z \in y))$
(Read: For any x, y, and z: z is in the intersection of x and y if and only if z is in x *and* z is in y.)

If we want to represent things with venn diagrams, we can think of intersection as the *overlap* of two sets:



The overlap (intersection) contains all the things contained in *both* circles (sets).

- If B is the set of all books and G is the set of things weighing 500g, then the intersection $B \cap G$ is just the set of books which weigh 500g. **Example 3**
- If D is the set of all dogs and G is the set of all gerbils weighing 500g, then the intersection $D \cap G$ is the set things which are both dogs and gerbils weighing 500g. **Example 4**

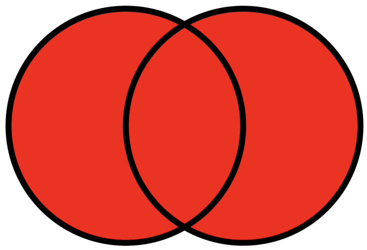
Note: The intersection of two sets does not have to be non-empty. In Example 4, $D \cap G$ is the set of all things which are dogs and gerbils weighing 500g, i.e., it contains nothing. We write this as either '{}' or '$\varnothing$'.

- For any set x, is the intersection $x \cap x$ distinct from x? **Question 2**
  - ⋆ No, $x = x \cap x$. Every member of x is trivially a member of both x and x.
    Thus, every member of x is a member of $x \cap x$. By extensionality, $x = x \cap x$.
- For any three sets, x, y, and z, is $(x \cap y) \cap z$ the same as $(x \cap z) \cap y$?
  - ⋆ Try and prove this yourself!

2.2. For any two sets x and y, there is a set called the *union*. The union of x and y is the set containing all the things which are *either* members of x or members of y. We write the union as $x \cup y$.

> **UNION**: $\forall x \forall y \forall z(z \in x \cup y \leftrightarrow (z \in x \vee z \in y))$
> (Read: For any x, y, and z: z is in the union of x and y if and only if z is in x *or* z is in y.)

If we want to represent this with venn diagrams, we can think of the union of two sets as follows:



The union contains *everything* in either circles (sets).

- If R is the set of all read books in the library and and U is the set of all unread books in the library, then the union $R \cup U$ is just the set of all books in the library. **Example 5**
- If D is the set of all dogs and G is the set of all gerbils weighing 500g, then the union $D \cup G$ is the set things which are either dogs or gerbils weighing 500g. **Example 6**

2.3. Another important relation between sets is so-called *inclusion* or *subsethood*. We say of two sets x and y that x is included in y, or x is a subset of y, just in case all the members of x are members of y. The inclusion of one set in another should not be confused with the membership of one set in another. If a set x is included in (or a subset of) set y, we write $x \subseteq y$.

> **INCLUSION/SUBSET**: $\forall x \forall y((Sx \wedge Sy) \rightarrow (x \subseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y))$
> (Read: For any sets x and y, x is a subset of y if and only if every member of x is a member of y.)

If we want to represent inclusion with venn diagrams, think of one circle (subset) inside another another (set).

- The set of all brown cats B is a subset of the set of all cats C: every member of B is a brown cat and so is a cat and so is a member of C. **Example 7**
- The set of all the people P is not a subset of the set of all the things I love L. There are some people (things in P) whom I do not love (things not in L). **Example 6**

2.4. We should distinguish inclusion from *proper inclusion*. A set x is properly included in y (or a proper subset of y) just in case every member of x is a member of y *and* there are members of y which are not members of x. We write $x \subset y$ for x is a proper subset of y. For instance the set of all brown cats is a proper subset of the set of all cats: there are non-brown cats. Proper inclusion differs from inclusion because a set x can be a subset of y even if $x = y$; but no set x can be a proper subset of itself.

3. Naive Comprehension.

3.1. So far, the laws governing sets have been quite straightforward: extensionality, intersection, union, subset, and proper subset. Each of these laws tell us how sets *relate* to one another. But no law we have discussed so far tells us *which* sets *exist*. Earlier, I said that there are all sorts of sets: the set of all cats, the set containing my left pinky finger, the Empire State Building and nothing else, and so on. Just as we have precise laws like extensionality, it would be nice to have a law which told us precisely which sets exist.

3.2. One such putative law is the so-called comprehension principle, or the *naive comprehension principle*. Take any sentence and make it an *open sentence*, i.e., remove one or more singular terms and replace them all with x. The results are what we called *conditions* like 'x is tall' or 'John does not like x or x does not like John'. Naive comprehension states that for any condition ϕ, there's a set containing the things satisfying ϕ.

> **NAIVE COMPREHENSION:** For any condition ϕ, there is a set y: $\forall x(\phi x \leftrightarrow x \in y)$

3.3. This principle certainly seems to be driving at the kind of idea we started with: sets are just a certain kind of collection and *anything you like* can be gathered together into a set. So, there's a set of all tall things because naive comprehension says that there is a set of all and only the things which satisfy 'x is tall'. There's a set consisting of the Empire State Building and my left pinky finger and nothing else because the naive comprehension principles tells us that there is a set of all and only the things which satisfy:

$$x = \text{Empire State Building or } x = \text{my left pinky finger}$$

Intuitive enough! The problem is that naive comprehension cannot be right because it is in fact *inconsistent*.

3.4. This is shown by **Russell's Paradox**. By naive comprehension, we know that some sets contain *themselves* and some do not. For instance, by comprehension, there is the set of sets which are non-empty ($\phi =$ 'x is a set with at least one member'). This set contains itself. But, by comprehension, there's the set of sets containing at least one cat (ϕ = 'x is a set and for some cat y: $y \in x$). This set is not a member of itself.

3.5. But what about *the set of all sets which do not contain themselves*? Naive comprehension tells us that there is such a set: it is the set of all and only the x such that $x \notin x$. Let's call this set R:

> **SET *R* 'THE RUSSELL SET'**: Set R is such that $\forall x(x \in R \leftrightarrow x \notin x)$.

But now paradox looms. Ask yourself: is R a member of itself? Either it is or it isn't:

(1) Suppose R is not a member of itself ($R \notin R$). This means that R is an x such that $x \notin x$. But R is the set of *all* sets x such that $x \notin x$. In which case, then, R *is* a member of itself ($R \in R$).
   - Mini Conclusion: If $R \notin R$, then $R \in R$.

(2) But now suppose that R is a member of itself ($R \in R$). This means that then R is an x such that $x \notin x$. (Why? Because R is just the set of sets which are not members of themselves.) In which case, by assuming that R is a member of itself, we come to the conclusion that it is *not* a member of itself.
   - Mini Conclusion: So if $R \in R$, then $R \notin R$.

So, naive comprehension tells us that there is a set—the set of all and only the sets which are not members of themselves, i.e., R—and R is a member of itself *if and only* if it is not a member of itself, i.e., $R \in R \leftrightarrow R \notin R$. That's a contradiction. So, naive comprehension cannot be right, it's inconsistent.