Sets, Relations and Probability. Part IA Formal Methods.

Lecture II, *More Basic Set Theory*, 20th February.

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Last lecture, we introduced the idea of a set, some basic notation and then talked about some laws governing sets like the axiom of extensionality, as well as the notions of intersection and union. Today, we'll look at some further set-theoretic ideas, particularly the notion of a powerset, an ordered n-tuple, and a cartersian product. Then we'll start to think about how we can use set theory to think about properties and relations.

1. Powerset

1.1. If x is a set, then **the powerset of** x, which we write $\mathcal{P}(x)$, is **the set of all subsets of** x. For every set x, there is the power set of x. To express the definition of the powerset of x it is useful to have some more notation for describing sets. Last week, we saw that if a set is finite, we can denote the set by enclosing a list naming the members of that set in set-brackets, i.e., $\{a, b, c\}$ is the set containing a, b, and c. But we can also write the set of Fs as $\{x \mid Fx\}$, i.e., the set of all and only x such that Fx. We define the powerset of x:

> **POWERSET:** $\mathcal{P}(x) = \{y \mid y \subseteq x\} = \{y \mid \forall z(z \in y \rightarrow z \in x)\}$
> (Read: The powerset of x is the set of all y such that y is a subset of x.)

1.2. Crucially, $\mathcal{P}(x)$ is the set of all *subsets* of x and thus the only members of $\mathcal{P}(x)$ are themselves sets. That means that the members of $\mathcal{P}(x)$ should not generally be confused with the members of x, although sometimes both $\mathcal{P}(x)$ and x *might* share members. Some examples will help illustrate this.

- If $x = \{1, 2\}$, then $\mathcal{P}(x) = \Big\{\{1\}, \{2\}, \{1, 2\}, \varnothing\Big\}$ **Example 1**
  That is, each of $\{1\}, \{2\}, \{1, 2\}$ and $\varnothing$ are subsets of $\{1, 2\}$ and nothing else is.
  It might seem weird that $\varnothing \in \mathcal{P}(\{1, 2\})$. But note that $\forall z(z \in \varnothing \rightarrow z \in \{1, 2\})$ is trivially true.
  Note: no $y \in \mathcal{P}(x)$ is a $y \in x$ or *vice versa*.
- If $x = \{1, \varnothing\}$, then $\mathcal{P}(x) = \Big\{\{1\}, \{\varnothing\}, \{1, \varnothing\}, \varnothing\Big\}$ **Example 2**
  Importantly, $\{\varnothing\}$ and $\varnothing$ are distinct—only the former is a non-empty set.
  Note: $\varnothing \in x$ and $\varnothing \in \mathcal{P}(x)$ and so x and its powerset in this case share a member.

1.3. Generally, $\mathcal{P}(x)$ has strictly more members than x. This is often phrased as the result that, for any set x, $\mathcal{P}(x)$ has a greater *cardinality* than x and is known as **Cantor's Theorem.** If some set x has n many members, for some natural number n (e.g., 0, 1, 2, 3, etc.), then the powerset has 2^n members. This fact is useful for checking whether you have found all the subsets of a given set. ("Have I got all 2^n subsets?")

1.4. Cantor's Theorem is not restricted to finite sets. Even if x is infinite, $\mathcal{P}(x)$ has a great cardinality than x. This is certainly a *very* interesting result, since it means that there are *different sizes of infinity*—some infinities are larger than others—and there is no largest cardinality, e.g., the infinite set of real numbers (e.g., 0, 1, 5.3, $\sqrt{2}$, etc.) is larger than the infinite set of the integers, i.e., positive or negative natural numbers.

2. Ordered Pairs

2.1. The **ordered pair** of a, b is written as $\langle a, b\rangle$. Importantly, the ordered pair $\langle a, b\rangle$ is not the same as the ordered pair $\langle b, a\rangle$, nor the set $\{a, b\}$. Crucial to the definition is that the *order matters*. We define this:

> **ORDERED PAIR**: The ordered pair $\langle a, b\rangle$ is the set $\{a, \{a, b\}\}$.

Thus, $\langle a, b\rangle = \{a, \{a, b\}\}$ and $\langle b, a\rangle = \{b, \{a, b\}\}$. By the Axiom of Extensionality, $\{a, \{a, b\}\}$ is not identical to $\{b, \{a, b\}\}$, since $a \in \{a, \{a, b\}\}$ and $a \notin \{b, \{a, b\}\}$ and so $\langle b, a\rangle \neq \langle b, a\rangle$, just as we want.

2.2. There are not just ordered *pairs*, but ordered *triples*, ordered *quadruples*, all the way up to **ordered n-tuples**, for any natural number n. We define an ordered n-tuple $\langle a_1, ..., a_n\rangle$ in the following way.

> **ORDERED *N*-TUPLE:** $\langle a_1, ..., a_n\rangle = \langle\langle a_1, ..., a_{n-1}\rangle, a_n\rangle$, for any natural number n.

2.3. This is a bit complicated and it's initially difficult to see how we use this definition. Here's an example.

> - How would we unpack the definition of $\langle a, b, c\rangle$? **Question 1**
>
>   (i) $\langle a, b, c\rangle = \langle\langle a, b\rangle, c\rangle = \{\langle a, b\rangle, \{\langle a, b\rangle, c\}\}$.
>
>   (ii) $\langle a, b\rangle = \{a, \{a, b\}\}$. Putting (i) and (ii) together: $\{\{a, \{a, b\}\}, \{\{a, \{a, b\}\}, c\}\}$
>
>   ⋆ **Note**: You need to know this definition exists, but usually when talking about ordered n-tuples, it suffices to just write them as $\langle a_1, ..., a_n\rangle$ and nothing more complicated.

3. Cartesian Products and Complements

3.1. With the notion of an ordered pair, we can define the **cartesian product** of two sets. For any two sets, y and z, there is the so-called cartesian product, $y \times z$. This is another set which contains all and only the ordered pairs $\langle a, b\rangle$, where the first of the pair is a member of y and the second of the pair is a member of z.

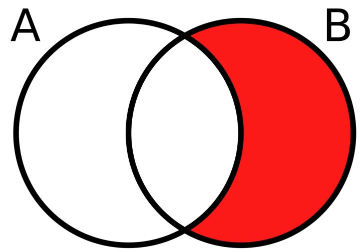
> **CARTESIAN PRODUCT**: $y \times z = \{\langle a, b\rangle \mid a \in y \wedge b \in z\}$
>
> (Read: The cartesian product $y \times z$ is the set of ordered pairs $\langle a, b\rangle$ where a is in y and b is in z.)

The cartesian product of a set y with itself, i.e., $y \times y$, is simple denoted y^2.

3.2. Another important idea is the **complement of two sets**. For any two sets, x and y, there is a set which contains all of the objects in x which are not in y. This is the *complement of y in x* and we write it as $x - y$.

> **COMPLEMENT OF *Y* IN *X***: $x - y = \{z \mid z \in x \wedge z \notin y\}$
>
> (Read: The complement of y in x contains any z which is in x but not in y.)

If we have two set A and B, we can visualise the complement of A in B in terms with a Venn diagram.



The complement of A in B is the red portion.

4. Expressing Yourself Set-Theoretically

4.1. We now have quite a number of set-theoretic tools (intersection, union, powerset, cartesian product, complements) which we can use to give expressions for a variety of different sets. You should be able to express sets using the range of different notation we've been using so far. Here are some examples.

Example 3

- Let S be the set of sandwiches sold in the Tesco meal deal, C be the set of crisps sold, and D be the set of drinks. We can then express:
  (i) All products sold in the meal deal: $(S \cup C) \cup D$
  (ii) All sandwich and crisps combinations: $S \times C$
  (iii) All vegetarian sandwich and crisps combinations: $(S - \{x \mid x \in S \text{ and contains meat}\}) \times C$.

4.2. You also should be able to use the definitions to give explicit examples of sets with certain features.

Question 2

- Give an example of the following sets:
  (i) **A set with exactly two members**.
    - Straightforward: $\{1, 2\}$
  (ii) **Two sets, the complement of one in the other has exactly two members**.
    - Straightforward: $\{1, 2, 3\} - \{1\} = \{2, 3\}$.
  (iii) **A set, the powerset of which has exactly two members**.
    - Remember that $\mathcal{P}(x)$ has 2^n members, if x has n members. So any set which has exactly one member, has a powerset which has exactly two members.
      For example, $\mathcal{P}(\{1\}) = \{\{1\}, \varnothing\}$. (Any set with exactly one member will do.)
  (iv) **Two sets, the cartesian product of which has exactly two members**.
    - If x has n members and y has m members, then $x \times y$ has n multiplied by m members.
      So we need a set x with one member and a set y with two, e.g., $\{1\}$ and $\{1, 2\}$.
      $\{1\} \times \{1, 2\} = \{\langle 1, 1\rangle, \langle 1, 2\rangle\}$

5. Properties, Relations, and Set Theory

5.1. It is standard to think that properties and relations are *intensional*, i.e., there can be two distinct properties which are had by exactly the same objects. The classic example: *being a renate* (having a kidney) and *being a cordate* (having a heart). That being said, it can be useful to sometimes only care about their extensions. The extension of a property is just all and only the objects which have that property.

5.2. We can represent the extension of a property as **the set of its instances**. So, the extension of *blue* is the set of all and only the blue things. Generally, then, the extension of P is the set $\{x \mid Px\}$. We can then represent various extensional relations between properties using sets. For example,

Example 4

- Two properties P and Q are **co-extensive** if and only if $\{x \mid Px\} = \{x \mid Qx\}$
- A property P **includes** a property Q if and only if $\{x \mid Px\} \subseteq \{x \mid Qx\}$

5.3. This might work for representing the extensions of properties, but it won't work for relations. Properties are exemplified by individual objects, but relations hold between multiple objects. Moreover, the order in which the objects are related is often important. Instead, then, we represent the extension of a relation with sets of ordered n-tuples. For a binary (two-place) relation like Rxy, the extension is given by a **special set of ordered pairs**: the extension of Rxy is $\{\langle x, y\rangle \mid Rxy\}$. For example,

Example 5

- The extension of x is taller than y is $\{\langle x, y\rangle \mid x \text{ is taller than } y\}$.
  - $\langle$Andre the Giant, Christopher$\rangle \in \{\langle x, y\rangle \mid x \text{ is taller than } y\}$. Andre the Giant is taller than me.
  - $\langle$Christopher, Andre the Giant$\rangle \notin \{\langle x, y\rangle \mid x \text{ is taller than } y\}$. I'm not taller than Andre the Giant.

5.3. Two final facts worth noting. First, the empty set $\varnothing$ is an extension of some relations: empty relations which don't relate anything to anything. For example, the relation of not being self-identical:

$$\text{The extension of not being self-identical} = \{\langle x, x\rangle \mid \neg x = x\} = \varnothing$$

Second, whilst we need ordered *pairs* to specify the extension of *two*-place relations, we need ordered triples for three-place relations, ordered quadruples for four-place relations, and so on. Generally, then, we give the extension of an n-place relation with a special set of n-tuples. For example,

$$\text{The extension of } x \text{ is taller than both } y \text{ and } z = \{\langle x, y, z\rangle \mid x \text{ is taller than both } y \text{ and } z\}$$