Sets, Relations and Probability. Part IA Formal Methods.

Lecture III, *Relations*, 23rd February.

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In the last two lectures, we outlined some basic set theory: it's notation, some principles governing sets, and some key notions, e.g., intersection, union, subset, powerset, cartesian product, and complement. At the end of the last lecture, we began to talk about relations. In this lecture, we'll discuss relations and their characteristics in more detail.

1. Relations

1.1. Relations are like properties in that certain things exemplify them, or certain things have that relation applied to them. Loosely, relations differ from properties in that relations are exemplified by more than one object: things stand in relation to other things. For instance, there is the property x is tall, but there is the relation of x is taller than y. The latter holds between x and y, whereas the former holds only of x. This is of course *loosely speaking*, however: some relations relate a thing to itself, e.g., the relation x is identical to x.

1.2. A relation is defined on a domain—the set of objects which the relation relates, or is applied to. We are not always interested in only one specific domain. We might, for instance, consider a relation, defined on the domain of cats, or defined on the domain of dogs, or people. The point is that when classifying relations as having certain features, we always do so relative to a domain. As you'll see, various ways of classifying relations are defined using quantifiers. Those quantifiers range over the domain the relation is defined on.

2. Simple Theory of Relations.

2.1. The *theory of relations* classies relations in various ways. The basic categories two-place relations can fall into are *reflexive*, *symmetric*, and *transitive*. A relation is reflexive if and only if for any object in the domain, the relation relates that object to *itself*. That is to say:

> **REFLEXIVE**: Relation Rxy is reflexive on a domain if and only if $\forall x Rxx$.
>
> * **Note**: Here, as stressed, $\forall x$ ranges over the domain.

A simple reflexive relation is x *is identical to* y. This relation is reflexive on any domain. Less trivial examples of reflexive relations would be x *was born in the same place as* y on the domain of all people. That is, everyone (in the domain of people) was born in the place where they were born. So,

$$\forall x(x \text{ was born in the same place as } x)$$

Of course, not all relations are reflexive: x *loves* y is not reflexive on the domain of people because not everyone loves themselves. So, $\neg\forall x(x \text{ loves } y)$ given the domain is all people.

2.2. A relation is *anti-reflexive* on a given domain if and only if for no object in that domain does the relation relate that object to itself. Note a relation is not anti-reflexive if $\neg\forall x Rxx$, it is not reflexive in that case. Rather:

> **ANTI-REFLEXIVE**: Relation Rxy is anti-reflexive on a domain if and only if $\forall x \neg Rxx$.

A simple anti-reflexive relation is x *is distinct from* y. Since identity is reflexive, there is no x such that x *is distinct from* x. This is anti-reflexive on every domain, i.e., $\forall x\neg(x \text{ is distinct from } x)$ given any domain.

2.3. A relation R is *symmetric* on a domain if and only if whenever R relates x to y in the domain, R also relates y to x. That is, if R is symmetric and relates two objects in the domain, the 'reverse' also holds.

> **SYMMETRIC**: Relation Rxy is symmetric on a domain if and only if $\forall x \forall y(Rxy \rightarrow Ryx)$.

Again, a simple example of a symmetric relation is identity: if $x = y$, then $y = x$, for any x and y. Identity is symmetric on any domain. A less trivial symmetric relation would be x *is biologically related to* y on the domain of all people. If x is biological related to y, then y must also be biologically related to x. So,

$$\forall x \forall y(\text{If } x \text{ is biologically related to } y, \text{ then } y \text{ is biologically related to } x)$$

Of course, not every relation is symmetric. Rather sadly, the relation of x loves y is not symmetric on the domain of all people: there are some people who love others, who do not love them back.

2.4. A relation R is anti-symmetric on a given domain if and only if two objects in the domain are related by R only if the 'reverse' does not hold. Again, this is not the same as just not being symmetric. Rather:

> **ANTI-SYMMETRIC**: Relation Rxy is anti-symmetric on a domain if and only if $\forall x \forall y(Rxy \rightarrow \neg Ryx)$.

A simple example of an anti-symmetric relation would be x *is the father of* y on the domain of all people. That is, if x in the domain is the father of y, then y cannot be the father of x.

2.5. A relation R is *transitive* on a domain if and only if for any three objects in the domain (x, y, z): if R relates x to y and R relates y to z, then R relates x to y. More precisely:

> **TRANSITIVITY**: Relation Rxy is transitive on a domain if and only if $\forall x \forall y \forall z((Rxy \wedge Ryz) \rightarrow Rxz)$.

Obvious transitive relations include the relation of identity, again: if $x = y$ and $y = z$, then $x = z$. Identity is transitive on any domain. Less trivial examples of transitive relations would include x *is an ancestor of* y or x *is in the same logic class* as y. Of course, not all relations are transitive, e.g., the relation of x loves y is not transitive on the domain of all people: I don't love everyone who the people I love love.

2.6. A relation R is *anti-transitive* on a domain if and only if for any three objects in the domain (x, y, z): if R relates x to y and R relates y to z, then it is not the case that R relates x to z. More precisely:

> **ANTI-TRANSITIVITY**: Relation Rxy is *anti-transitive* on a domain iff $\forall x \forall y \forall z((Rxy \wedge Ryz) \rightarrow \neg Rxz)$

Again, anti-transitivity is not just failing to be transitive. The latter holds if $\neg \forall x \forall y \forall z((Rxy \wedge Ryz) \rightarrow Rxz)$.

2.7. A relation R is an *equivalence relation* on a given domain if and only if R is reflexive, symmetric and transitive. Spelling this out in full detail:

> **EQUIVALENCE**: Relation Rxy is an *equivalence relation* on a domain if and only if all of the following:
> (i) $\forall x Rxx$ (Reflexive)
> (ii) $\forall x \forall y(Rxy \rightarrow Ryx)$ (Symmetric)
> (iii) $\forall x \forall y \forall z((Rxy \wedge Ryz) \rightarrow \neg Rxz)$ (Transitive)

Again, an obvious example of an equivalence relation is identity. This is an equivalence relation on every domain. Another, less trivial example, of an equivalence relation would be *x and y are in the same logic class* on the domain of Part IA Philosophy students. Many relations are not equivalence relations: a relation only has to be either not reflexive, not symmetric, or not transitive to fail to be an equivalence relation.

2. More on Domains and Relations.

2.1. At the outset, I stressed that relations can be classified in these above ways only relative to domains. This is very important. For instance, earlier, I noted that the relation *x loves y* is not reflexive on the domain of people because some people don't love themselves. However, if we defined the relation *x loves y* on a domain of people where those who don't love themselves were excluded, the relation *x loves y* would be reflexive. The characteristics of relations are sensitive to their domains.

1.6. Another point worth stressing is that some relations have their characteristics trivially. For instance, a relation is symmetric on a domain iff $\forall x \forall y(Rxy \rightarrow Ryx)$ is true given that domain. But $\forall x \forall y(Rxy \rightarrow Ryx)$ can be *trivially* true if $\forall x \forall y \neg Rxy$, i.e., if no two things in the domain are related by R. Thus, although it may seem *surprising*, the relation x i*s a baby born after baby y* is symmetric on the domain of babies born at the same time t. The same applies for transitivity, if we have a relation R where $\forall x \forall y \forall z \neg(Rxy \wedge Ryz)$.

2.2. It should be noted as well that the characteristics of relations, i.e., whether they are reflexive, symmetric, or transitive, depends on what *actually exists*, not on what could have existed. Some relations, therefore, have their characteristics given a certain domain only contingently, e.g., *x loves y*. Others have their characteristics necessarily. For instance, the relation of identity is necessarily reflexive, symmetric and transitive.

2.3. A final point worth emphasising: any open sentence with x and y free is a relation—an open sentence being a sentence containing one or more free variables. This means that we can take any sentence and replace two singular terms with the free variables x and y and generate a two-place relation. A relation like *x won the 1979 election and y won the 1997 election* only holds between Margaret Thatcher and Tony Blair and in that order; but it is still a relation. The broader point: any set of ordered-pairs defines some relation.

3. Classifying and Giving Examples

3.2. You should also be able to give examples of relations with different characteristics on a given domain.

Give an example of the following relation, explaining your answer. **Question 1**

(i) **A reflexive, symmetric, but not transitive relation.**

Model Answer. Consider *x is either in the same logic class or discussion group as y*. Domain: Part IA Philosophy Students. It is **reflexive**: x is either in the same logic class or discussion group as x. It is **symmetric**: if x is either in the same logic class or discussion group as y, then y is either in the same logic class or discussion group as x. It is **not transitive**: x and y can be in the same logic class or discussion group, y and z can be in the same logic class or discussion group, but that doesn't mean that x and z are in the same logic class or group.

(ii) **A symmetric, transitive, but not reflexive relation**. *Left as an exercise for you!*

(iii) **A reflexive, transitive, but not symmetric relation.**

(iv) **A relation which neither reflexive, symmetric, nor transitive.**

Note: You should always make sure to specify the domain of your example relation.

3.2. You should also be able to classify example relations defined on a certain domain.

Say whether the following are reflexive, symmetric, or transitive on the given domain. **Question 2**

(i) x **is older than** y **on the domain of all living people.**

Model Answer. First, x is older than y is **not reflexive** on the domain of all living people because no individual in that domain is older than themselves. Second, x is older than y on the domain of all living people is **not symmetric** on the domain of all living people because if x is older than y, then y cannot be older than x. Third, x is older than y on the domain of all living people **is transitive** on the domain of all living people. If x is older than y and then y is older than z, then x must also be older than z. Since in all of these argument x, y and z were arbitrary members of the domain of all living people, x is older than y on the domain of all living people is not-reflexive, not-symmetric, but transitive.

(ii) x **is a gerbil and** y **has two legs on the domain of all people and gerbils.**

Model Answer. First, x is a gerbil and y has two legs is **not reflexive** on the domain of all people and gerbils, since not every member of the domain is a gerbil—there are people as well! Second, x is a gerbil and y has two legs is **not symmetric**. Counterexample: if a is a gerbil and b is a human with two legs, then the following is true.

$$a \text{ is a gerbil and } b \text{ has two legs} \tag{1}$$

(1) tells us: there are two things in the domain which satisfy x is a gerbil and y has two legs. But b is a *human* with two legs, so b is not a gerbil. So:

$$\neg(\ b \text{ is a gerbil and } a \text{ has two legs}) \tag{2}$$

(1) and (2) give us a counterexample to symmetry.

Third, x is a gerbil and y has two legs **is transitive**. Why? For convenience, just write Rxy for x is a gerbil and y has two legs. If $\neg Rxy$ for arbitrary x and y, then R is transitive trivially. So, next suppose Rxy, for arbitrary x and y. It follows that x is a gerbil and y has two legs. This means that y could be either a human or a two-legged gerbil (the domain contains all the humans and gerbils). So, we have to consider both possibilities or 'Cases':

Case One (y **is a human)**: If y is a human, then Ryz is false, for any z. But that means that $(Rxy \wedge Ryz) \rightarrow Rxz$ is trivially true.

Case Two (y **is a two-legged gerbil)**: If y is a two-legged gerbil, then Ryz holds only if z has two legs. z either has two legs or not. If z has two legs, then Ryz *and* Rxz hold. So, $(Rxy \wedge Ryz) \rightarrow Rxz$ is true. If not, then Ryz is false. So, $(Rxy \wedge Ryz) \rightarrow Rxz$ is trivially true.

So, we argued from Rxy, $(Rxy \wedge Ryz) \rightarrow Rxz$, where x, y, z were arbitrary. So, R is transitive.

Note: To show a relation does not have a characteristic, you must specify a counterexample.