Sets, Relations and Probability. Part IA Formal Methods.

Lecture VII, *Repeated Events*, 8th March.

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Last week, we discussed conditional probability in more detail, focusing on two versions of Bayes' Theorem. This week, we'll further extend our discussion of probability and look at probabilities for repeated events.

1. Repetition with Independent Events

1.1. So far, we have been discussing how to calculate the probability of single events, e.g., the probability that a die lands on a one, or the probability that a die lands on a odd number. Often, though, we will want to calculate the probability of repetitions of those events. For instance, how would we calculate the probability that the die lands on a one at least once, if rolled twice?

1.2. How we calculate the probability of repeated events depends on whether the events are probabilistically independent or not. It is simpler when the events are independent, so we look at that first. Let's focus on a simple example: tossing a coin twice. Suppose we want to know the probability that the coin lands heads at least once. How would we calculate that? First, we need to define the relevant outcome space V.

Outcome Space Individually: If we were only interested in a single toss, the outcome space $V_i = \{H, T\}$, where H represents the outcome of the coin landing heads and T the coin landing tails.

But we are not interested in a single toss. That means that the relevant outcome space is not $\{H, T\}$.

Outcome Space Double: If we are interested in find the probability of certain events given that the coin is tossed twice, the relevant outcome space $V = \{\langle H, H\rangle, \langle H, T\rangle, \langle T, H\rangle, \langle H, H\rangle\}$. Here, e.g., $\langle H, T\rangle$ represents the outcome of the coin landing heads (H) first toss and then tails (T) second toss.

Two points worth noting. First, each element of V is an ordered-pair reflecting the fact that the order in which the coin lands a certain side matters, i.e., landing heads and then tails is a different outcome to landing tails and then heads. Second, note that the relevant outcome space V for the double toss is just V_i^2, i.e., $V_i \times V_i$, where V_i is the outcome space for the *single toss*. For a repeated trial of independent events, e.g., coin toss, the outcome space is $V^2 = \{\langle x, y\rangle | x \in V \wedge y \in V\}$, where V is the outcome space for the single trial.

1.3. Back to our question: what's the probability that the coin lands heads at least once, if tossed twice? To work out this, we need to identify the relevant *event* representing the coin landing heads at least once. The outcome space is $V = \{\langle H, H\rangle, \langle H, T\rangle, \langle T, H\rangle, \langle H, H\rangle\}$. The relevant event: $\{\langle H, H\rangle, \langle H, T\rangle, \langle T, H\rangle\}$. Note:

$$\{\langle H, H\rangle, \langle H, T\rangle, \langle T, H\rangle\} = \{\langle H, H\rangle\} \cup \{\langle H, T\rangle\} \cup \{\langle T, H\rangle\} \tag{1}$$

This means that $Pr(\{\langle H, H\rangle, \langle H, T\rangle, \langle T, H\rangle\}) = Pr(\{\langle H, H\rangle\} \cup \{\langle H, T\rangle\} \cup \{\langle T, H\rangle\})$. Moreover, since $\{\langle H, H\rangle\} \cap \{\langle H, T\rangle\} \cap \{\langle T, H\rangle\} = \varnothing$, we can then use Axiom 3 (Lecture IV):

$$Pr(\{\langle H, H\rangle\} \cup \{\langle H, T\rangle\} \cup \{\langle T, H\rangle\}) = Pr(\{\langle H, H\rangle\}) + Pr(\{\langle H, T\rangle\}) + Pr(\{\langle T, H\rangle\}) \tag{2}$$

Since the coin is fair, we take the probability of each event to be $1/4$ and thus the probability of each singleton event containing exactly one outcome $1/4$. Thus: $Pr(\{\langle H, H\rangle, \langle H, T\rangle, \langle T, H\rangle\}) = 3/4$.

2. Repetition with Dependent Events

2.1. Matters are slightly more complicated with repeated trials involving dependent events. An example of a repeated trial involving dependent events would be calculating probabilities of certain cards been drawn from a pack when two cards are drawn *without replacement*. Again, let's focus on a specific example. Suppose we want to calculate the probability that at least one of two card draws *without replacement* is a seven.

2.2. First, we need to define the relevant outcome space V. For a single card draw, the outcome space V is the 52-member set, one member of V for each card in the pack. However, unlike in the case of repeated trials involving independent events, we cannot take the outcome space for the two card draw to just be V^2, e.g., V^2 would include $\langle 9H, 9H\rangle$, where $9H$ is the outcome of drawing the nine of hearts—$\langle 9H, 9H\rangle$ is thus the event of drawing the nine of hearts *twice* which is impossible, since we were drawing without replacement. Instead, the outcome space represents all the two draw outcomes, removing the 'double draws':

$$V^2 - \{\langle x, x\rangle \,|\, x \in V\} \tag{3}$$

Our outcome space for the repeated draw contains 52×51 members, since $\{\langle x, x\rangle \,|\, x \in V\}$ has 52 members. Since we are drawing purely at random, we assign each outcome an equal probability of $(\frac{1}{52 \times 51})$

2.3. Back to the question: what is the probability that at least one of the two card draws *without replacement* is a seven? There are three relevant events we are interested in:

$A :=$ the event of the first card being a seven, but the second card not.

$B :=$ the event of the second card being a seven, but the first card not.

$C :=$ the event of both cards being a seven.

Each of A, B, C are strictly speaking subsets of ordered pairs in the outcome space; but here we can ignore this detail since things would get too cumbersome. What we need to calculate is $Pr(A \cup B \cup C)$. Since $A \cap B \cap C = \varnothing$, $Pr(A \cup B \cup C) = (Pr(A) + Pr(B) + Pr(C))$. So, we need to calculate $Pr(A), Pr(B), Pr(C)$:

$Pr(A)$ and $Pr(B)$. A and B both have 4×48 members. So, $Pr(A) = Pr(B) = \dfrac{4 \times 48}{52 \times 51} = \dfrac{192}{2652}$

$Pr(C)$. C has 4×3 members. So, $Pr(C) = \dfrac{4 \times 3}{52 \times 51} = \dfrac{12}{2652}$

So $Pr(A \cup B \cup C) = (Pr(A) + Pr(B) + Pr(C)) = \dfrac{33}{221}$.

3. Conditional Probabilities of Repeated Trials.

3.1. We can also calculate the *conditional* probabilities of repeated trials. Suppose we are tossing a fair coin three times. What's the probability that all the tosses land heads, given that at least one of them lands heads? First, we define our outcome space. This time we need the following set of ordered triples:

$$V = \{\langle H, H, H\rangle, \langle H, H, T\rangle, \langle H, T, H\rangle, \langle H, T, T\rangle, \langle T, H, H\rangle, \langle T, H, T\rangle, \langle T, T, H\rangle, \langle T, T, T\rangle\}$$

Here, we assign each outcome an equal probability of $1/8$. Letting A be the event of all tosses are heads and B be the event that at least one toss is heads, we want to calculate $Pr(A|B)$. That is:

$$Pr(A|B) = \frac{Pr(\{\langle H, H, H\rangle\} \cap \{\langle H, H, H\rangle, \langle H, H, T\rangle, \langle H, T, H\rangle, \langle H, T, T\rangle, \langle T, H, H\rangle, \langle T, H, T\rangle, \langle T, T, H\rangle\})}{Pr(\{\langle H, H, H\rangle, \langle H, H, T\rangle, \langle H, T, H\rangle, \langle H, T, T\rangle, \langle T, H, H\rangle, \langle T, H, T\rangle, \langle T, T, H\rangle\})}$$

This quickly reduces to the following, since the numerator is just $Pr(\{\langle H, H, H\rangle\})$:

$$Pr(A|B) = \frac{Pr(\{\langle H, H, H\rangle\})}{Pr(\{\langle H, H, H\rangle, \langle H, H, T\rangle, \langle H, T, H\rangle, \langle H, T, T\rangle, \langle T, H, H\rangle, \langle T, H, T\rangle, \langle T, T, H\rangle\})}$$

Applying Axiom 3 and the fact that the probabilities we assigned to each outcome were equal: $Pr(A|B) = \frac{1}{7}$.

3.2. We can also calculate the conditional probabilities of repeated trials involving dependent events. What is the probability that if two cards are drawn from a pack without replacement, both cards drawn are sevens given that at least one them is a seven? Let C be the event of both cards drawn being a seven and let A be the event of at least one card drawn being a seven. From the example in (2.3), we know that:

$$Pr(C) = \frac{12}{52 \times 51} = \frac{1}{221} \text{ and } Pr(A) = \frac{33}{221}$$

So, we calculate the probability that both cards drawn without replacement are sevens given that at least one of them is seven, i.e., $Pr(C|A)$ as follows, since $C \cap A = C$.

$$Pr(C|A) = \frac{Pr(C \cap A)}{Pr(A)} = \frac{Pr(C)}{Pr(A)} = \frac{(1/221)}{(33/221)} = \frac{1}{33}$$

4. Conditional Probability is Sensitive

4.1. Earlier, we calculated the probability of a coin tosses three times landing heads each time, given that it landed heads at least once. We showed that this was $1/7$. What about the probability that a coin tossed three times lands heads each time, given that it lands heads *the first time*? In this case, the relevant events:

- The coin lands heads *the first time* $F = \{\langle H, H, H\rangle, \langle H, H, T\rangle, \langle H, T, H\rangle, \langle H, T, H\rangle\}$
- The coin lands heads *each time* $E = \{\langle H, H, H\rangle\}$

Therefore, the probability that the coin tossed three lands lands heads each time, given that it lands heads the *first time* is calculated as follows, assuming each outcome is given equal probability of $1/8$.

$$Pr(E|F) = \frac{Pr(\{\langle H, H, H\rangle\})}{Pr(\{\langle H, H, H\rangle, \langle H, H, T\rangle, \langle H, T, H\rangle, \langle H, T, H\rangle\})} = \frac{1/8}{4/8} = \frac{1}{4}$$

4.2. At first glance, this is a strange result. The probability that a coin lands heads each time, given that it lands heads at least once is $1/7$. But the probability that a coin lands heads each time, given that it lands heads *the first time* is $1/4$. So, the chance of the coin landing heads each time is nearly doubled, if it lands heads the first time. But why should that matter? The important observation: there are fewer outcomes where the coin lands heads *first* compared to outcomes where the coin lands heads *at least once*. So, we have the same numerator in both calculations, but different denominators.

The General Lesson: Conditional probabilities are *highly sensitive* to what you are conditionalising on. Always do the calculations thoroughly—intuition is a bad guide to probability.