Sets, Relations and Probability. Part IA Formal Methods.

Lecture IV, *More on Relations*, 23rd February.

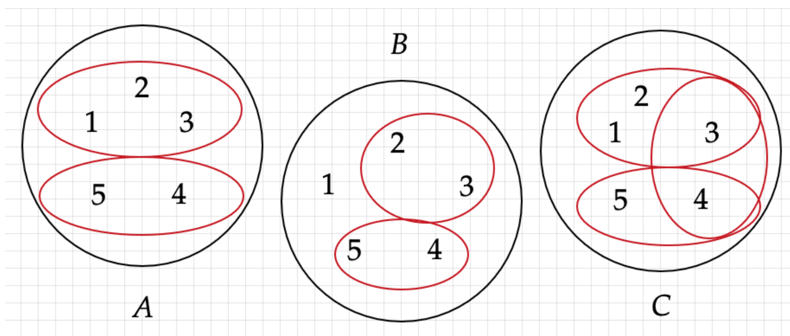
Christopher J. Masterman (cm789@cam.ac.uk, christophermasterman.com)

Last lecture, we outlined some basic characteristics we can ascribe to relations—reflexivity, anti-reflexivity, symmetry, anti-symmetry, transitivity, and anti-transitivity—as well as some basic facts about relations. This lecture, we'll go a little further in characterising relations.

1. Equivalance Relations, Classes, and Partitions.

1.1. Recall that a binary relation R is an **equivalence relation** on a domain if and only if R is reflexive, symmetric, and transitive. A simple example of an equivalence relation on a domain is the relation of identity on the domain: the set of ordered pairs $\langle x, x \rangle$, where x is an element of the domain. A less trivial example would be the relation of x and y are in the same logic class on the domain of all Part IA Philosophy students.

1.2. An important concept is the notion of a **partition of a set**. Take a set x. Informally, a partition of x is a way of dividing up the content of x into regions in such a way that everything in x is in exactly one region. In other words, it is a way of dividing up a set into **non-overlapping** regions which 'cover' the entire set. To illustrate, suppose we have a set $\{1, 2, 3, 4, 5\}$ and three 'ways' of dividing up its elements, A, B, and C:



A represents a partition; B and C do not.

Crucially, B misses an element out and C carves the set up in overlapping ways. A partition must divide the set up **exhaustively** and leave no element out and **exclusively** and have each element in only one region.

1.3. Formally, we define a partition P on a set X as follows. P is a set of subsets of X such that:

PARTITION: $\forall z\Big(z \in X \rightarrow \underbrace{\exists p(p \in P \wedge z \in p}_{\text{(i)}} \wedge \underbrace{\forall q(z \in q \wedge q \in P \rightarrow q = p))}_{\text{(ii)}}\Big)$

(Read: For any element z of X, (i) there exists a subset p of X in the partition P and z is in p and (ii) for any other subset q of X in the partition P, z is in q only if q and p are the same subset.)

Here, the first part of the formal definition (i) secures that a partition is **exhaustive** and the second part of the formal definition (ii) secures that a partition is **exclusive**.

1.4. Now, there's an interesting connection between partitions on sets and equivalence relations. For any equivalence relation R on a domain, there is a special partition on that domain. Each of the individual sectors of that partition are known as **equivalence classes**. That special partition are the members of the domain which relate to each other by R. Let's illustrate this connection with some examples.

1. Let the domain (set) be the set of all Part IA Philosophy students S. Consider the relation x **is in the same logic class as** y. This is an equivalence relation: reflexive, symmetrical, and transitive. As such, we can define the set of equivalence classes of S defined by this relation:

   $$\{p \subset S \mid \forall x \forall y(x, y \in p \leftrightarrow x \text{ is in the same logic class as } y))\}$$

   This set of equivalence classes is a special partition P (set of subsets $p \subset S$) on the set of all Part IA Phil. students. Crucially, any student in S is in some $p \in P$ and no student is more than one $p \in P$.
2. Let the domain (set) be the set of all people who have ever lived. The relation x **was born at the same time as** y is an equivalence relation. The equivalence classes of the set of all people who have ever lived, then, are just the subset of people all born at t_1, the subset of people all born at t_2, and so on...

1.5. As I have stressed a number of times, identity is an equivalence relation. What do the set of equivalence classes defined by this relation look like for any set? Well, these equivalence classes will simply be **singletons**. So, if $X = \{1, 2, 3, 4\}$, the set of all the equivalence classes in X corresponding to identity will just be $\{\{1\}, \{2\}, \{3\}, \{4\}\}$. You can see that this satisfies the definition of a partition on X above. The notion of a partition and equivalence classes has some interesting philosophical applications. Russell and Frege proposed to define numbers by identifying them as the equivalence classes of sets which are equinumerous.

2. Other Characteristics of Relations

2.1. For any relation R, there is a relation known as the **converse** or **inverse** of R. This is often denoted as R' (pronounced: "R prime"). The converse/inverse of a relation R is the relation which holds between y and x if R holds between x and y. More precisely, in symbols we would say that :

> **CONVERSE/INVERSE**: For any relation R there is a converse/inverse R' such that $\forall x \forall y(Rxy \leftrightarrow R'yx)$.

For example, if R is the relation x **is directly to the north of** y, then R' is x **is directly to the south of** y. Similarly, if R is the relation of x **is smaller than** y, then R' is the relation of x **is bigger than** y. There is a nice relationship between R' and symmetry: R is its own inverse/converse R' iff R is symmetric.

2.2. For any relation R, there is the relation known as the ancestral of R. This is often written as R^*.

> **ANCESTRAL**: The ancestral R^* of R is such that R^*xy iff for some $z, w, v, ..., u$: Rxz, Rzw, Rwv, Ruy.

An obvious example of an ancestral relation is the relation of x **is an ancestor of** y as the ancestral relation of x **has** y **as a parent**. Someone y is my ancestor just in case there are some people $z, w, u, ..., v$ such that I have z as a parent, z has w as a parent, and so on, ... all the way to v has y as a parent. In Part IA Personal Identity, we looked at the ancestral of x **remembers being** y to reformulate the memory criterion of personal identity over time from Reid's objection that this criterion entails that identity is non-transitive.

3. Functions

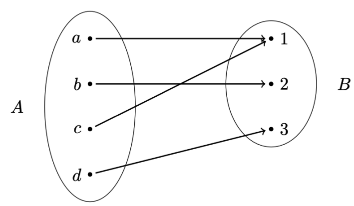
2.3. A relation R is a function on a domain if and only if it relates each member of the domain to some unique other individual. In other words, for any three objects x, y, z, if R relates x to y and R relates x to z, then y is just the same thing as z. More precisely, we can define this as follows.

FUNCTION: Relation R is a **function** iff $\forall x \forall y \forall z((Rxy \wedge Rxz) \rightarrow y = z)$.

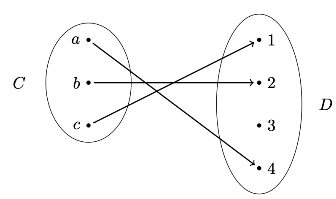
Now, you may well have come across functions presented with notation like '$f(x) = y$' and '$g(y) = x$' and may not be used to thinking about them as a special kind of relation. But remember what I emphasised last week: any set of ordered pairs can be taken to represent a relation. Similarly, sets of ordered pairs are also typically taken to represent functions. So functions and relations, as construed here, go hand in hand.

2.4. There are three key properties of functions that you ought to know: what it means for a function to **surjective**, **injective** and **bijective**. To understand these, it's easier to think of functions as relating two sets and to adopt the notation of $f, f(x), g(y) = x$, etc. The first set, we call the **domain** of the function, and the second set we call the **range**. A function takes some element x from the domain ($x \in$ *domain*) and **maps it to** an element of the range ($y \in$ *range*). If x is mapped to y by f, were write $f(x) = y$. Of course, f is still a function and so each element of the domain is mapped to only **one** element of the range.

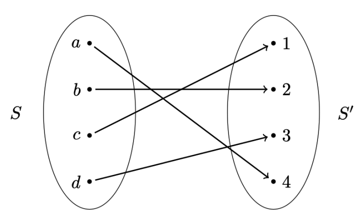
2.5. A function is **surjective** iff, for every element in the range y, there is some element in the domain x and $f(x) = y$. It makes sure that every element in the range is paired with at least element of the domain:



2.6. A function is **injective** just in case, for any x and y in the domain, if $f(x) = f(y)$, then $x = y$. An injective function does not map distinct members of the domain to the same element in the range:



2.7. Finally, a function is **bjiective** if and only if it is injective *and* surjective. This is often known as a one-to-one correspondence. If a function is bijective, then every element of the domain is mapped to a unique element of the range. Diagrammatically, with domain as S and range as S':



2.8. A final point. An interesting application of functions is that they can allow us to express claims about the equinumerosity of sets without a prior appeal to the concept of number. Two sets are equinumerous if and only if there is a bijection between two sets. Thus, the Russell-Frege proposal of understanding numbers as equivalence classes of equinumerous sets is not egregiously circular.