Propositional Contingentism and Possible Worlds[*][†]

Christopher J. Masterman

Abstract

Propositional contingentism is the view that what propositions there are is a contingent matter—certain propositions ontologically depend on objects which themselves only contingently exist. Possible worlds are, loosely, complete ways the world could have been. That is to say, the ways in which everything in its totality could have been. Propositional contingentists make use of possible worlds frequently. However, a neglected, but important, question concerns whether there are any notions of worlds which are both theoretically adequate and consistent with propositional contingentism. Some notion of a possible world is adequate if the systematic connection between, at least, possibility and truth at some possible world holds. Here, I argue that no adequate notion of a possible world is available to at least those who subscribe to one natural formulation of propositional contingentism. I also show that this result contrasts with a simple and adequate definition of a possible world available to the necessitist—those who hold that necessarily everything necessarily exists.

According to contingentism, there might have been things which might have been nothing. For instance, it strikes many as undeniable that, had my parents never met, I would not have existed—in such a case, I would have been nothing. In fact, some contingentists go further and argue that had I never existed the proposition [I do not exist] would not *itself* exist. That is to say, in such cases, there just would be no content to the claim that I exist or that I do not exist, see (Adams, 1981), (Fine, 1985), (Fitch, 1996), (Prior, 1967), (Speaks, 2012), and (Stalnaker, 2012). For the sake of simplicity, let *propositional contingentism* just be the view that some propositions and some non-propositions are contingent, where propositions are contingent *because* they depend, for their existence, on the existence of other contingent objects.

[*]This is final draft of 'Propositional Contingentism and Possible Worlds' in *Synthese*. Please cite the published version: https://link.springer.com/article/10.1007/s11229-022-03879-5

[†]I'd especially like to thank Peter Fritz for many helpful discussions about many versions of this paper and I'd like to thank Franz Berto, Greg Restall, Thomas Randriamahazaka, and Wouter Cohen for helpful discussions on various drafts of this paper.

Propositional contingentists often make use of possible worlds—maximally specific ways the world could have been. For instance, Robert Stalnaker in describing a consequence of his view writes that '[t]he singular proposition *Socrates does not exist* is a proposition that will be true of ... only possible worlds in which that proposition does not exist' (Stalnaker, 2012: 46). Another example is Jason Turner, who formulates one common version of propositional contingentism in terms of possible worlds:

> If a proposition p exists and predicates something of an object a, then in any in any possible world W, if a does not exist in W, then p also does not exist in W. (Turner, 2005: 191)

Such talk is ubiquitous, but it prompts a natural although neglected question: to what extent is talk of possible worlds available to the propositional contingentist at all? Of course, some uses of possible worlds are quite innocent, e.g., the nature of 'worlds' in a model-theoretic semantics can be somewhat arbitrary. However, worlds are also used in philosophical accounts of modality. Here, I am concerned with this second use. To think that worlds play a significant philosophical role in our account of modality is here understood to at least involve a commitment to the truth of the right sort of biconditional connecting modality and possible worlds, where the latter are entities understood as genuine ways, or specifications of the ways, the world could have been.

I argue that, for the necessitist—one who thinks that necessarily everything, propositions and non-propositions alike, necessarily exists—possible worlds suitable for playing this role can be defined. However, I argue that the contingentist is unable to replicate this success. My strategy throughout the paper is to investigate the question of whether different views can make use of possible worlds by looking at whether we can embed adequate notions of possible worlds into logics for those views. Here, I take a logic for a metaphysical view to be a proof system which has as theorems only truths which hold *generally* and *necessarily* according to that view. Developing these logics is not intended to tell us what these modal metaphysical views should think about *logic* in a grand sense of the word. Rather, such logics are intended to allow us to more rigorously assess what a view is, and can be, committed to.[1] In this framework, a natural measure for whether a certain view can make use of possible worlds is to follow Menzel and Zalta (2014) and say that a notion of a possible world in such a logic is *adequate*

[1]Compare (Williamson, 2013) which focuses on modal logics that capture the *metaphysically universal* truths of modality, e.g., 'We want a theory of metaphysical modality that consists of all the sufficiently general truths about it.' (Williamson, 2013: 92).

only if there is a theorem in that logic which can be read as stating a biconditional connecting metaphysical possibility and necessity to truth in some, or all, worlds, respectively, i.e., the Leibnizian biconditionals. I assume, here, that regardless of whether some view takes worlds to feature in an analysis, or reduction, of modality, or whether world-facts ground modal-facts, or so on, the Leibnizian biconditionals should be generally and necessarily true, according to that view. A view *can* make use of possible worlds, if a definition of a world can be given in its logic and such biconditionals are theorems.

Here's how this paper will proceed. In §1, I start with the simpler case and develop a necessitist logic $\vdash_n$ and show that the systematic connections between possibility as truth at some world and necessity as truth at all worlds, for at least one definition of a possible world, are theorems of $\vdash_n$. Then, in §2, I develop a weaker proof system $\vdash_c$ which captures a natural and promising form of propositional contingentism. I show that the availability of contingentist possible worlds is inconsistent in $\vdash_c$ with some well-motivated claims about possibly indistinguishable entities that the propositional contingentist should accept. Thus, they cannot articulate adequate definitions of possible worlds (§3). I then consider an extension of $\vdash_c$ with an actuality operator $\vdash_c^{@}$ to investigate whether the contingentist can define worlds for which the connection between possibility and truth at some world holds only *actually*, if not necessarily (§4). I show that this requirement is equally problematic for the contingentist. Finally, in §6, I prove some formal results underpinning my argument.

Before turning to these arguments, it is worth first saying something in the way of motivating the *first-order* treatment of propositions in this paper, since this contrasts with recent work on propositional contingentism.[2] Here, propositions are understood as objects over which first-order variables can range. In contrast to this, propositional contingentism is reasonably well-studied in higher-order settings, notably in the work of Fritz (2016, 2017, 2018a, 2018b) and Fritz and Goodman (2016, 2017). This, in turn, is part of a growing trend of applying the resources of higher-order logic to metaphysical questions, particularly to investigate intensional entities like propositions and properties.[3] Now, this paper is not the place to address questions about whether such a trend is on the whole worthwhile. However, it is worth noting that the use of higher-order resources in metaphysics is not widely accepted and taking this approach

[2]Thank you to a reviewer for noting the need for this motivation.

[3]See (Skiba, 2021) for an overview of, and (Fritz and Jones, forthcoming) for an excellent collection of papers discussing, this trend.

is far from mandated.[4] Moreover, a large portion of the recent work on propositional contingentism has been done, assuming a relational type theory, particularly (Fritz and Goodman, 2016) and (Fritz, 2018a, 2018b), and thus it is of a broader theoretical interest to investigate questions about propositional contingentism independently of the specific logical and metaphysical constraints intrinsic to such a setting.[5] This, in my view, warrants exploring questions about propositional contingentism from a first-order perspective.

1 Necessitism and Possible Worlds

I begin by outlining the simpler case: a necessitist logic $\vdash_n$ in which we can investigate whether the necessitist can define an adequate notion of a possible world. Outlining the simpler case allows me to discuss principles which carry over to the contingentist logic $\vdash_c$. Moreover, I will show that the necessitist *can* define an adequate notion of a world, showing that any failure of a contingentist project to make sense of worlds is not the product of a general poverty in the very notion of a possible world.

1.1 The Necessitist Logic $\vdash_n$

A simple option for a necessitist is to take possible worlds as maximally consistent pluralities of propositions. The necessitist proof system $\vdash_n$ is a plural, two-sorted, first-order modal logic which I will extend to include propositional abstraction and a truth predicate, in a language I call $\mathcal{L}^W$. $\mathcal{L}^W$ is composed of the following lexicon. First, for each natural number *n*:

- Singular non-propositional variables, x_n, y_n, z_n.
- Singular propositional variables, p_n, q_n, r_n.
- Plural non-propositional variables, xx_n, yy_n, zz_n.
- Plural propositional variables pp_n, qq_n, rr_n.

[4] For recent, general scepticism of higher-order resources applied in metaphysics, see (Menzel, forthcoming) and (Pickel, forthcoming). For an extensive early formal development of first-order theories of propositions, see (Fine, 1980). See (Bealer, 1982, 1993, 1998) and (Bealer and Mönnich, 1989) for an influential defence of a unified first-order account of properties, relations and propositions. More recently, a first-order approach to propositions is defended in (Merricks, 2015).

[5] See (Menzel, 1993: 64–66) and (Bealer, 1994) for arguments *for* a type-free approach to intensional entities. For discussion of the expressive limits to type-theoretic settings, see (Linnebo, 2006: 154-156)

- Denumerably many *n*-place singular predicates, $R_n^1, R_n^2, ..., R_n^i$.
- Denumerably many *n*-place plural predicates, $\overline{R}_n^1, \overline{R}_n^2, ..., \overline{R}_n^i$.

We have the following function, logical connectives and logical predicate symbols:

- One function symbol, $\sim$.
- Singular identity symbol, $=$; an *is among* symbol, $\prec$; a truth predicate, T.
- $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$, $\Diamond$, $\Box$, (,) and [,].[6]

The syntax of $\mathcal{L}^W$ is specified as follows. First, the syntactic rules for complex singular terms in $\mathcal{L}^W$: first, $\sim t$ is a propositional singular term if t is a propositional singular term and second, $[\phi]$ is a singular term if ϕ is a formula.[7] The well-formed formulae $\phi \in \mathcal{L}^W$ are defined recursively. That is, first, $Rt_1, ..., t_n$ and $\overline{R}tt_1, ..., tt_n$ are formulae iff R is an *n*-place singular predicate and $t_1, ..., t_n$ are *n* singular terms of any sort and $\overline{R}$ is an *n*-place plural predicate and $tt_1, ..., tt_n$ are *n* plural terms of any sort, respectively. Second, if ϕ is a formula, v and vv are variables of any sort, then $\neg\phi$, $\Box\phi$ and $\Diamond\phi$, $\forall v\phi$, $\exists v\phi$, $\forall vv\phi$, and $\exists vv\phi$, are also all formulae of $\mathcal{L}^W$. Third, if t and tt are singular and plural term of any sort, then $t \prec tt$ is a formula. Finally, if ϕ and ψ are formulae, $\phi \vee \psi$, $\phi \wedge \psi$, $\phi \rightarrow \psi$, and $\phi \leftrightarrow \psi$ are formulae.

The necessitist logic $\vdash_n$ is the result of combining the axioms for classical quantification and the axioms for an S5 modal logic, extending this to include plural quantification and propositional abstraction. To define $\vdash_n$, we say, as is standard, that a $\mathcal{L}^W$ *substitution instance* of a well-formed formula of the language of propositional logic Φ is any expression which is the result of uniformly replacing every propositional variable in Φ with a well-formed formula of $\mathcal{L}^W$. Second, I will write $\ulcorner\phi[t/v]\urcorner$ for the result of uniformly replacing all free instances of v in ϕ with *t*—here *v* and *t* are either both a singular or plural variable and term of any sort.[8] Third, we say that a term *t* is *free for* variable *x* just in case no free occurrence of *x* in ϕ lies within the scope of a quantifier $\forall y$ or $\exists y$, where *y* is a variable which is free in *t*.

[6] All parts of the logical vocabulary are primitive so as to not bake into any theory non-obvious propositional identities such as $[\phi \wedge \psi] = [\neg(\neg\phi \vee \neg\psi)]$.

[7] '*t*' is a meta-variable for propositional singular terms, i.e. p, q, r or $[\phi]$.

[8] Note that the square bracket notation in the metalanguage used to indicate substitution should be kept distinct from the square brackets used to indicate abstraction from formulae to propositions. Since formulae with free variables can be enclosed by square brackets, I should also note that substitution is applied uniformly in, and out, of the scope of the square brackets, e.g. if $\phi := \mathrm{T}[Fx] \rightarrow Fx$, then $\phi[t/x] := \mathrm{T}[Ft] \rightarrow Ft$.

Definition 1 **($\vdash_n$)** *Let $\vdash_n$ be the proof system in $\mathcal{L}^W$ consisting of the following principles. Here ϕ and ψ are wff of $\mathcal{L}^W$, v stands for any singular variable of any sort, vv, vv' for any distinct plural variables of any sort, and $\boldsymbol{v}$ for any variable, singular or plural, of any sort. Unless stated otherwise, take v, vv, and vv' to be of the same sort—either all propositional, or all non-propositional. The following are axioms:*

(PC) *Any $\mathcal{L}^W$ substitution instance of a tautology.*

($\forall$1) $\forall \boldsymbol{v}\phi \to \phi[t/\boldsymbol{v}]$*, provided t is free for $\boldsymbol{v}$ in ϕ.*

(QE) $\exists \boldsymbol{v}\phi \leftrightarrow \neg\forall \boldsymbol{v}\neg\phi$.

(Comp) $\exists v\phi(v) \to \exists vv\forall v(v \prec vv \leftrightarrow \phi(v))$*, for ϕ free of vv.*

(NE) $\forall vv\exists v(v \prec vv)$.

(Ext) $\forall vv\forall vv'(\forall v(v \prec vv \leftrightarrow v \prec vv') \to (\phi(vv) \leftrightarrow \phi(vv')))$.

(P1) $t \prec tt \to \Box t \prec tt$*, for any terms t, tt.*

(I1) $t = t$*, for any term t.*

(I2) $t = t' \to (\phi[t'/v] \leftrightarrow \phi[t/v])$*, for any t, t' free for v in ϕ.*

(T[]) $\mathrm{T}[\phi] \leftrightarrow \phi$.

(PNeg1) $\neg\mathrm{T}t \leftrightarrow \mathrm{T}{\sim}t$*, for any propositional term t.*

(PNeg2) $t = [\phi] \leftrightarrow {\sim}t = [\neg\phi]$*, for any propositional term t.*

(OE) $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$.

(K) $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$.

(T) $\Box\phi \to \phi$.

(5) $\Diamond\phi \to \Box\Diamond\phi$.

The following are rules of inference:

($\forall$2) $\phi \to \psi \,/\, \phi \to \forall \boldsymbol{v}\psi$*, provided $\boldsymbol{v}$ is not free in ϕ.*

(MP) $\phi, \phi \to \psi \,/\, \psi$.

(N) $\phi \,/\, \Box\phi$

Some comments are in order. Pluralities, here, are thought to be *nothing over and above* the objects which are among them. They are simply the things, considered together in some plural fashion. (Comp), (NE), (Ext) and (P1) govern the behaviour of such pluralities.[9] Strictly speaking, (P1) is independent of the natural extension of the theory of plural quantification into a theory of modal plural quantification (Hewitt, 2012). However, I take it to be a reasonable modal axiom governing plurals, even if it is not quite wholly uncontentious.[10] The conception of pluralities as nothing over and above the things among them also motivates the idea that pluralities are individuated extensionally and this allows us to define a notion of plural identity.

Definition 2 (Plural Identity) *Let* $\ulcorner tt = tt' \urcorner$ *abbreviate* $\ulcorner \forall v(v \prec tt \leftrightarrow v \prec tt') \urcorner$*, where* v*,* tt *and* tt' *are variables and terms, respectively, each of the same, but any, sort.*

Thus, plural parallels for the singular identity axioms are theorems. In other words, for any terms tt, tt': $\vdash_n tt = tt'$ and $\vdash_n tt = tt' \rightarrow (\phi[tt/xx] \leftrightarrow \phi[tt'/xx])$. The latter theorem follows from (2) and (Ext)—the axiom that extensionally equivalent pluralities satisfy the same open-sentences of $\mathcal{L}^W$.

Several axioms of $\vdash_n$ govern propositions and, although it is controversial what propositions are, such axioms in $\vdash_n$ characterise propositions minimally. First, (T[]) captures the relation between propositions and formulae of $\mathcal{L}^W$—$[\phi]$ is the proposition expressed by ϕ, and so, naturally, $[\phi]$ is true if and only if ϕ. Second (PNeg1) and (PNeg2) together capture one way in which propositions relate to each other in ways which mirror the relations between sentences indicated by sentential operations. We should at least like to say that, for any proposition p, there is a distinct proposition which is the *negation* of p. Since propositional quantification in the logic here is nominal, I need propositional analogues to the sentential operators. For my purposes, I need only be explicit about negation–we have $\sim$ as a function which takes any proposition as argument, and returns that proposition's negation. Of course, we also need some comprehension principle for propositions, i.e., a principle to govern the conditions under which a proposition exists. In $\vdash_n$, the quantification is classical and so the comprehension principle is derived:

[9]For more discussion of (Comp), (NE), and (Ext) see (Linnebo, 2017: §1.2).

[10]See (Linnebo, 2016) and (Uzquiano, 2011) for arguments *for* (P1) and also (Hewitt, 2015) for arguments *against* (P1)

Theorem 3 (Comprehension) $\vdash_n \exists p(p = [\phi])$, *for* $\phi \in \mathcal{L}^W$.

It is thus guaranteed that there is a proposition, i.e., $[\phi]$, for every formula $\phi \in \mathcal{L}^W$. The classical quantification, identity axioms and the rule of necessitation also guarantee the truth of singular and plural necessitism, for both non-propositions and propositions alike in $\vdash_n$. Thus: $\vdash_n \Box\forall x\Box\exists y(y = x)$, $\vdash_n \Box\forall p\Box\exists q(q = p)$, $\vdash_n \Box\forall xx\Box\exists yy(yy = xx)$, and $\vdash_n \Box\forall pp\Box\exists qq(pp = qq)$. One final feature of $\vdash_n$ should be noted, namely that the Barcan Formula is a theorem, i.e. $\vdash_n \Diamond\exists x\phi \rightarrow \exists x\Diamond\phi$. From hereon, I will refer to this as 'BF'.

1.2 Possible Worlds in $\vdash_n$

Possible worlds are supposed to be maximally specific ways the world could have been and there are several ways of implementing this idea. The strategy taken in $\vdash_n$ is to take possible worlds as maximally consistent pluralities of propositions. Of course, maximally consistent pluralities of propositions are not the *only* way of defining possible worlds. For instance, we have the idea that worlds are maximally consistent sets of propositions (Adams, 1981); maximally inclusive and possible states of affairs (McMichael, 1983)(Plantinga, 1976)(Plantinga, 1979); maximally consistent individual propositions (Fine, 1977a)(Stalnaker, 2012); or certain special properties of the world (Forrest, 1986)(Stalnaker, 2012). Pluralities rather than sets are used here in order to avoid worries about cardinality, see (Bringsjord, 1985), (Grim, 1986), (Menzel, 1986), and (Menzel, 2012).[11] But, all in all, very little, if anything, of what follows involves making assumptions too specific to worlds as pluralities of propositions or making assumptions for which there would not be some analogous assumptions about salient features of alternative conceptions of possible worlds. (I will discuss this point about the generality of the arguments presented in this paper in more detail in §3.)

To start, then, we first define what it means for some propositions to be *maximal* and some propositions to be *consistent*:

Definition 4 (Maximality): *Propositions pp are* maximal *if* $\forall p(p \prec pp \vee {\sim}p \prec pp)$.

[11] Given such an expressive language like $\mathcal{L}^W$ with unrestricted abstraction, plurals, and a truth predicate, there ought to be a background worry about consistency. A consistency proof of $\vdash_n$ and $\vdash_c$ in §6 is provided to answer this worry.

Definition 5 (Consistency): *Propositions pp are* consistent *if* $\Diamond\forall p(p \prec pp \to \mathrm{T}p)$.

We then define a *world*:

Definition 6 (World): *Propositions pp are a* world ($\mathrm{W}pp$) *just in case pp are both* maximal *and* consistent.

As possible worlds, such pluralities ought to have certain features. Importantly, some propositions are true, and some propositions are false, relative to a world. Following the notation in (Menzel and Zalta, 2014), I write $\ulcorner pp \models p \urcorner$ to stand for the claim that p is true relative to pp. In $\vdash_n$ this is defined as follows, where I write $\ulcorner \overline{\mathrm{T}}pp \urcorner$ for $\ulcorner \forall p(p \prec pp \to \mathrm{T}p) \urcorner$.

Definition 7 (Truth In): *Let some proposition p be* true in *some propositions pp* ($\ulcorner pp \models p \urcorner$) *just in case* $\Box(\overline{\mathrm{T}}pp \to \mathrm{T}p)$.

The definitions here allow for a simple and attractive view of possible worlds. Simple, since we require no more than (4)–(7) and the uncontroversial characterisation of propositions in $\vdash_n$. Attractive, since the systematic connections between possibility and necessity, on the one hand, and truth relative to some or all worlds, respectively, on the other hold as theorems in $\vdash_n$. Let's call these systematic connections the *Fundamental Theorems of Possibility and Necessity*, following (Menzel and Zalta, 2014).

Theorem 8 (Fundamental Possibility): $\vdash_n \Diamond\phi \leftrightarrow \exists pp(\mathrm{W}pp \wedge pp \models [\phi])$, *for any formula $\phi \in \mathcal{L}^W$ with no free 'pp'*.

Theorem 9 (Fundamental Necessity NECESSITY): $\vdash_n \Box\phi \leftrightarrow \forall pp(\mathrm{W}pp \to pp \models [\phi])$, *for any formula $\phi \in \mathcal{L}^W$ with no free 'pp'*.

The proof of (8) and (9) to follow is related to the proof in (Menzel and Zalta, 2014: 345–348). My proof requires four preliminary lemmas—one about possibility and possible truth, one about propositions, and two about worlds. Here, *t* is any singular propositional term.

Lemma 10: *For any formula* $\phi \in \mathcal{L}^W$: $\vdash_n \Diamond \mathrm{T}[\phi] \leftrightarrow \Diamond\phi$.

Lemma 11: *Let* $\ulcorner\tau(pp)\urcorner$ *abbreviate* $\ulcorner\forall p(p \prec pp \leftrightarrow \mathrm{T}p)\urcorner$. *The following hold.*

(i) $\vdash_n \mathrm{T}t \rightarrow \exists pp(\tau(pp) \wedge t \prec pp)$

(ii) $\vdash_n \tau(pp) \rightarrow \mathrm{W}pp$

Lemma 12: $\vdash_n \mathrm{W}pp \rightarrow \Box\mathrm{W}pp$

Lemma 13: $\vdash_n \mathrm{W}pp \rightarrow (pp \models t \leftrightarrow t \prec pp)$.

Now, here's a sketch of how we derive (8) and (9). First we use (10) to show that, if there are some propositions which qualify as a world and $[\phi]$ is true in those, then $\Diamond\phi$ must hold. This establishes the right-to-left direction of (8)—if there exists a world w and $[\phi]$ is true in w, then $\Diamond\phi$. The left-to-right direction of (8) is more demanding. First, we show that if $\Diamond\phi$ then, by (11)(i), possibly some propositions are such that $\tau(pp)$ and $[\phi] \prec \tau(pp)$, i.e., $[\phi]$ is possibly among some propositions which are all and only the true propositions. Second, by (11)(ii), we know that, necessarily, if pp are such that $\tau(pp)$, then pp are a world. By (13), this establishes that if $\Diamond\phi$ holds, then *possibly* there is a world in which $[\phi]$ is true. Given the assumption that propositions necessarily exist and metaphysical modality satisfies the principles of S5, it then follows that there *is* such a world. (9) follows from (8) as a corollary. Formally, the proofs are as follows.

Theorem 8 (Fundamental Possibility): $\vdash_n \Diamond\phi \leftrightarrow \exists pp(\mathrm{W}pp \wedge pp \models [\phi])$, *for any formula* $\phi \in \mathcal{L}^W$ *with no free 'pp'.*

Proof. First, the right-to-left direction.[12]

(1) $\vdash_n \exists pp(\mathrm{W}pp \wedge pp \models [\phi]) \rightarrow \exists pp(\Diamond\overline{\mathrm{T}}pp \wedge \Box(\overline{\mathrm{T}}pp \rightarrow \mathrm{T}[\phi]))$ D.6, D.7

(2) $\vdash_n \exists pp(\Diamond\overline{\mathrm{T}}pp \wedge \Box(\overline{\mathrm{T}}pp \rightarrow \mathrm{T}[\phi])) \rightarrow \Diamond\mathrm{T}[\phi]$ QML

(3) $\vdash_n \exists pp(\mathrm{W}pp \wedge pp \models [\phi]) \rightarrow \Diamond\phi$ (1), (2), L.10

[12]For convenience I typically omit stating applications of (MP) and (PC). Here 'D.*x*' = 'Definition *x*', 'L.*x*' = 'Lemma *x*', 'T.*x*' = 'Theorem *x*', and '(*x*)' = 'Line (*x*) of the present proof'.

Second, the left-to-right direction.

(1) $\vdash_n \Diamond\phi \to \Diamond\exists pp(\tau(pp) \wedge [\phi] \prec pp)$ L.10, L.11(i), (N)

(2) $\vdash_n \Diamond\exists pp(\tau(pp) \wedge [\phi] \prec pp) \to \Diamond\exists pp(\mathrm{W}pp \wedge [\phi] \prec pp)$ L.11(ii), (N)

(3) $\vdash_n \Diamond\exists pp(\mathrm{W}pp \wedge [\phi] \prec pp) \to \Diamond\exists pp(\Box\mathrm{W}pp \wedge \Box[\phi] \prec pp)$ L.12, (P1), (N)

(4) $\vdash_n \Diamond\exists pp(\Box\mathrm{W}pp \wedge \Box[\phi] \prec pp) \to \exists pp\Diamond(\Box\mathrm{W}pp \wedge \Box[\phi] \prec pp)$ (BF)

(5) $\vdash_n \exists pp\Diamond(\Box\mathrm{W}pp \wedge \Box[\phi] \prec pp) \to \exists pp(\Diamond\Box\mathrm{W}pp \wedge \Diamond\Box[\phi] \prec pp)$ QML

(6) $\vdash_n \exists pp(\Diamond\Box\mathrm{W}pp \wedge \Diamond\Box[\phi] \prec pp) \to \exists pp(\mathrm{W}pp \wedge [\phi] \prec pp)$ S5

(7) $\vdash_n \Diamond\phi \to \exists pp(\mathrm{W}pp \wedge [\phi] \prec pp)$ (1)–(6)

(8) $\vdash_n \Diamond\phi \to \exists pp(\mathrm{W}pp \wedge pp \models [\phi])$ (7), L.13

□

Theorem 9 (Fundamental Necessity): $\vdash_n \Box\phi \leftrightarrow \forall pp(\mathrm{W}pp \to pp \models [\phi])$, *for any formula* $\phi \in \mathcal{L}^W$ *with no free 'pp'.*

Proof. First, the left-to-right direction.

(1) $\vdash_n \forall pp(\mathrm{W}pp \to ([\phi] \prec pp \vee [\neg\phi] \prec pp))$ (PNeg2), D.4

(2) $\vdash_n \exists pp(\mathrm{W}pp \wedge \neg(pp \models [\phi])) \to \exists pp(\mathrm{W}pp \wedge pp \models [\neg\phi])$ (1), L.13

(3) $\vdash_n \exists pp(\mathrm{W}pp \wedge pp \models [\neg\phi]) \to \Diamond\neg\phi$ T.8

(4) $\vdash_n \Box\phi \to \forall pp(\mathrm{W}pp \to pp \models [\phi])$ (2), (3), (QE), (OE)

Second, the right-to-left direction.

(1) $\vdash_n \Diamond\neg\phi \to \exists pp(\mathrm{W}pp \wedge pp \models [\neg\phi])$ T.8

(2) $\vdash_n \exists pp(\mathrm{W}pp \wedge pp \models [\neg\phi]) \to \exists pp(\mathrm{W}pp \wedge \neg(pp \models [\phi]))$ D.4, D.6, (PNeg2)

(3) $\vdash_n \Diamond\neg\phi \to \exists pp(\mathrm{W}pp \wedge \neg(pp \models [\phi]))$ (1), (2)

(4) $\vdash_n \forall pp(\mathrm{W}pp \to pp \models [\phi]) \to \Box\phi$ (3), (QE)

□

This completes the exposition of a simple, necessitist theory of possible worlds. The systematic connections between possibility, necessity and truth at some, or all, worlds, respectively, are theorems of $\vdash_n$. As noted in (Menzel and Zalta, 2014: 336), few philosophers explicitly argue for the connection between possibility, necessity and truth at a world. Thus, the previous results are interesting. In short, given the conception of pluralities in $\vdash_n$, the necessitist simply needs some propositions from which theoretically adequate possible worlds can be defined.[13] Thus, the necessitist does not even require a potentially suspect primitive notion of a possible world. Moreover, these results show that, in principle, there is no general poverty in the very notion of a world itself. The question for the rest of the paper then is whether we can do the same with weaker assumptions and formulate adequate worlds consistent with propositional contingentism.

2 Propositional Contingentism and Logic $\vdash_c$

Before I develop the propositional contingentist logic $\vdash_c$ and investigate whether we can embed theoretically adequate definitions of worlds into such a logic, I should first discuss the nature of propositional contingentism in more detail. Now, it is beyond the scope of this paper to survey all formulations of the view. Instead, I will first sketch one common and compelling view of contingently existing propositions. I will then discuss how we should go about axiomatizing this view to capture the general and necessary truths of propositional contingentism.

The view which interests me finds a clear expression in (Prior, 1957). His view is that, if some sentence ϕ contains a term, then ϕ expresses a proposition which exists only in the cases in which the referent of that term exists (Ibid., pg. 34). Of course, Prior holds this view for distinctly Priorian reasons: the contingency in what propositions there are arises, since he holds that to say that x exists is just to say that there are facts about x, where facts are, for Prior, true propositions (Ibid.). Nonetheless, this view—that propositions expressed by sentences featuring terms depend ontologically on the referents of those terms—has a wide appeal in the literature, e.g., see Jeff Speaks (2012: 529) and Jason Turner (2005: 192) for clear endorsements of this view.

[13]Strictly speaking, all the necessitist really needs is the existence of *some* entities which behave minimally as the "propositions" in $\vdash_n$ behave. As a reviewer noted, it's worth noting that the necessitist account of worlds here uses primitive modal notions, so if a *reduction* of modal-talk is the benchmark for success, this theory is not successful.

Now, how we go about constructing a logic for this view involves making several decisions. To begin with, it is, of course, a contingentist view. Since necessitism results as a combined effect of our quantifier and identity axioms—specifically (∀1) and (I1)—and the necessitation rule (N), we need to amend one or more of those principles. There are two standard approaches. The first is to restrict (N) so that not all theorems are necessary, see (Prior, 1957: 48–49), (Adams, 1981: 27), (Deutsch, 1990: 99), and (Menzel, 1991: 359). The second option is adopt a free logic, allowing for only a *restricted* rule of existential generalisation. This means that from $\vdash_c x = x$ we can only derive the rather trivial $\vdash_c x = x \rightarrow (\mathrm{E}x \rightarrow \exists y(y = x))$, where 'E' is an existence predicate. Thus we can only innocuously conclude that $\vdash_c \forall x \Box(\mathrm{E}x \rightarrow \exists y(y = x))$. (I will discuss the meaning of the existence predicates later in this section.)

In my view, the second option is preferable because we have the following good reasons for retaining (N). First, a minor reason: the few positive results for the contingentist presented in §3 show that (N) is crucial to securing some of the positive results for the propositional contingentist and so restricting (N) only further impoverishes $\vdash_c$. This being said, however, the crucial reason concerns the role of the logic $\vdash_c$. To emphasise again, I am interested in axiomatizing truths which hold generally and *necessarily* for the propositional contingentist. In other words, the formulae derivable in $\vdash_c$ are only those formulae which remain true for the contingentist when prefixed with necessity operators and universal quantifiers in any sequence. Thus, the set of theorems of $\vdash_c$ should be closed under necessitation—this requires us to adopt (N) as a rule in $\vdash_c$. It is worth stressing here, however, that I do not take accepting (N) in $\vdash_c$ to have any bearing on whether the set of theorems of some *one true modal logic*, if there is such a logic, to be closed under necessitation. This argument for (N) is particular to the role of $\vdash_c$ as a logic which captures only the general and necessary truths of propositional contingentism.

The logic $\vdash_c$ is, of course, a *propositional* contingentist logic. Thus we need a distinctive comprehension principle to capture the contingency of propositions. In contrast to $\vdash_n$, such a principle must be adopted as an explicit axiom in $\vdash_c$. Here's how we formulate this principle. Recall that the motivating thought was that propositions expressed by sentences featuring free variables or complex terms ontologically depend on the referents of those parameters. Thus, we can read off which propositions are contingent existents by looking at the syntax of the sentences expressing those propositions. Let's write that a sentence features *n* terms $t_1, ..., t_n$ by writing $\ulcorner \phi^{t_1,...,t_n} \urcorner$. Here, $t_1, ..., t_n$ can contain any sort of term, singular or plural. With this in mind, then, we adopt the following axiom scheme, where $\ulcorner \mathrm{E}/\bar{\mathrm{E}} \urcorner$ stands for the appropriate existence predicate.

(P) $\vdash_c \exists p(p = [\phi^{t_1,\ldots,t_n}]) \leftrightarrow (\mathrm{E}/\bar{\mathrm{E}}t_1 \wedge \ldots \wedge \mathrm{E}/\bar{\mathrm{E}}t_n)$

Notably, when $n = 0$, we take the right-hand side conjunction to be some tautology, i.e., if $\ulcorner \phi \urcorner$ contains no free variables or complex terms, then $\vdash_c \exists p(p = [\phi])$.

(P) is clearly not neutral with respect to all choice points in the metaphysics of propositions. The conception of propositions, for instance, is not *coarse-grained*, i.e., it is not the case that any two propositions with the same truth value in all worlds are identical, e.g., see (Stalnaker, 1976). The contingentist here individuates propositions both by their truth, *and* existence, conditions. However, note, that (P) *alone* does not determine how *fine* such a fine-grained individuation should be. That is, (P) is consistent with any view of individuation which holds that propositions are *at least* individuated by their truth, and existence, conditions. Now, in order to make the later negative arguments general, no further axioms of $\vdash_c$ concern the individuation of propositions.

I take it that the conception of contingently existing propositions so far outlined does not *alone* prevent the contingentist from also adopting the compelling principles (T[]), (PNeg1), and (PNeg2) as axioms. That is to say, we should still want every proposition to have a negation and that the truth of a proposition $[\phi]$ be tied to the truth of the sentence ϕ. However, it is important to discuss what we should think about serious actualism—the view that in order for an object to exemplify a property, or in this context satisfy a predicate, it must exist. Crucially, (T[]), (P), (N), contingentism and the requirement that a proposition must exist to be true are jointly inconsistent.[14] Of course, (P) is an essential element of our axiomatization of propositional contingentism. (N), as I stressed earlier, is also an essential element of $\vdash_c$, given the role $\vdash_c$ plays.[15] Thus, we must choose between (T[]) and serious actualism. In my view, we ought to reject serious actualism, even if some will view this to be a cost. The first point to note is that serious actualism is controversial, it being most notably rejected in (Pollock, 1985: 126–129), (Fine, 1985: 163–171) and (Salmon, 1987: 95). In fact, both Fine and Salmon explicitly

[14]Since $\vdash_c \Box(Fx \vee \neg Fx)$, it follows from (T[]), (N), and (UG) that $\vdash_c \forall x \Box \mathrm{T}[Fx \vee \neg Fx]$. Given (P), it then follows that $\forall x \Box \mathrm{E}x$.

[15]Several prominent arguments for restricting (N) appeal to serious actualism, see Prior (1957: 34), Adams (1981: 27), Deutsch (1990: 98), and Menzel (1991: 358–359). One may worry that an argument against serious actualism which *uses* (N) is circular. However, we have *independent* reasons to accept (N), as I outlined above, and so no circularity ensues. Thanks to a reviewer for noting this.

argue against the weaker idea that a proposition's truth requires its existence—an idea also rejected in (Mitchell-Yellin and Nelson, 2016: 1538). Moreover, it has also been noted how serious actualism in conjunction with propositional contingentism is deeply problematic, see (Fritz and Goodman, 2016: 655), and (Jacinto, 2019: 491–496).

Second, (T[]) is an intuitively compelling principle; but it is also a *powerful* principle when combined with (N), allowing us to derive the necessitated truth schema, i.e., $\Box(\mathrm{T}[\phi] \leftrightarrow \phi)$. In the present context what's important is that we afford the contingentist the strongest possible resources that are within the bounds of their contingentism, investigating whether they can make use of such resources to articulate adequate definitions of worlds. Serious actualism is not entailed by contingentism, nor is the weaker idea that simply the truth of a proposition implies its existence, and so we are free to develop a contingentism which rejects both. Indeed, rejecting it allows us to prove some key positive results for the contingentist, e.g., Theorem 22 below. Thus, in rejecting serious actualism, the contingentist is free to accept principles, like (T[]), which are intuitive and which allow the contingentist to preserve crucial logical resources utilised in developing the simple necessitist theory of worlds in §1.2: rejecting serious actualism puts them in a *prima facie* stronger position to secure the existence of adequate possible worlds.

Although many contingentists accept serious actualism in some form, it is beyond the scope of this paper to offer anything like a decisive argument *against* serious actualism.[16] We must make a choice regarding serious actualism, since it is deeply non-obvious that precise discussions of this kind can be done independently of the question of serious actualism. However, whilst rejecting serious actualism may be motivated in the present context, we should be mindful that the arguments which follow thus target only *one* family of ways of developing propositional contingentism and that the arguments can be blocked by a refusal to deny serious actualism. Of course, this goes without saying that blocking the arguments in this paper by accepting serious actualism is not alone sufficient to show that propositional contingentism is consistent with theoretically adequate worlds, let alone show that propositional contingentism and serious actualism constitute a viable package of views. However, it is also beyond the scope of this paper to do that work here.[17]

Now, a few final comments are in order, before I fully outline the contingentist logic. $\vdash_c$ is a *plural* modal logic and we have to be careful with how we think about pluralities in a contin-

[16]See (Stephanou, 2007) for a thorough article arguing *for* serious actualism.

[17]Thanks to a reviewer for emphasising the need for these caveats.

gentist setting. I have, at several points, utilised both a singular and plural existence predicate. Here, I define these as follows.

Definition 14: $\ulcorner \mathrm{E}t \urcorner$ *abbreviates* $\ulcorner \exists x(x = t) \urcorner$, *for distinct terms t and x*

Definition 15: $\ulcorner \bar{\mathrm{E}}tt \urcorner$ *abbreviates* $\ulcorner \exists xx(xx = tt) \urcorner$, *for distinct terms* tt *and* xx.

(15) requires that we have an identity sign for pluralities. In $\vdash_c$, the previous definition (2) will not suit.[18] Instead, in $\vdash_c$, we must take plural identity to be defined in terms of *necessary* coextensiveness: $\ulcorner tt = tt' \urcorner$ abbreviates $\ulcorner \Box \forall y(y \prec tt \leftrightarrow y \prec tt') \urcorner$. Another issue is that we need to more carefully tie down the natural thought that a plurality exists only in cases in which all of the objects among that plurality exist. Of course, a plurality here is nothing over and above those things among it. However, for similar reasons to those discussed with plural identity, we cannot hope to capture this natural idea with a quantificational expression in a free logic. Instead, we partially capture the idea with the following principle, where t and tt are arbitrary terms of any, but the same, sort.

(PE) $\bar{\mathrm{E}}tt \rightarrow (t \prec tt \rightarrow \mathrm{E}t)$ *is an axiom.*

Indeed, the same kind of issues arise with how we should think about a plural predicate for the collective truth of some propositions. In $\vdash_c$, the quantifier expression $\ulcorner \forall p(p \prec pp \rightarrow \mathrm{T}p) \urcorner$ only defines joint truth on the assumption that all propositions among some jointly consistent propositions necessarily co-exist and this evidently does not hold.[19] The proposal, then, is to extend $\mathcal{L}^W$ to a language $\mathcal{L}^W_{\bar{\mathrm{T}}}$ which includes a plural truth predicate $\bar{\mathrm{T}}$ as a primitive.[20] The

[18] If xx exists this cannot simply hold because there are some things yy and xx and yy have the same *existent* things among them. For instance, suppose xx to be the things *a*, *b* and *c* and yy just the things *a* and *b*. If *c* were not to exist, it would nonetheless be true that there *exist* some things, namely yy, and everything (in the world, so to speak) among xx, namely *a* and *b*, would be among yy.

[19] Consider some propositions *pp* counting among them only three propositions, $p_1 = [\exists x(x = y)]$, $p_2 = [\neg \exists x(x = y)]$, $p_3 = [\psi]$, where $[\psi]$ is a proposition which is both necessarily true and necessarily exists and y is some contingent individual. If y were not to exist p_1 and p_2 would not exist and $[\psi]$ would both exist and be true. In which case, '*all*' propositions among *pp* would be true.

[20] As a reviewer noted, one may worry about the availability of $\bar{\mathrm{T}}$, especially in light of 'the problem of incompossibles' as discussed in (Williamson, 2013) and (Fritz and Goodman, 2017b). It is of course difficult to *decisively* motivate the intelligibility of $\bar{\mathrm{T}}$, but it is worth noting that many plural predicates like $\bar{\mathrm{T}}$ are intelligible even in the absence of reductions to expressions featuring only singular predicates, e.g., xx successfully carry y. Moreover, since I go on to show, in §3, that the contingentist is unable to secure the existence of adequate possible worlds in $\vdash_c$ even with $\bar{\mathrm{T}}$, here I grant its intelligibility for the sake of argument.

following principle restricts the behaviour of the plural truth predicate and should be included in the contingentist logic, where t, t', and tt are any propositional terms.

($\bar{\mathrm{T}}$) $\bar{\mathrm{T}}tt \to (t \prec tt \to \mathrm{T}t)$ *is an axiom.*

Of course, without the ability to quantify over non-existent propositions, we are unable to express the sufficient condition for the truth of some pp, i.e., that *all* p among pp are true. But ($\bar{\mathrm{T}}$) restricts the plural truth predicate enough for what is required here.

We're now in a position to provide the following formal definition of the propositional contingentist logic $\vdash_c$ within which we can explore the availability of propositional contingentist possible worlds.

Definition 16 ($\vdash_c$): *Let $\vdash_c$ be the proof system in $\mathcal{L}^W_{\bar{\mathrm{T}}}$ consisting of the following principles, where $\boldsymbol{v}$ stands for any variable, plural or singular of any sort, v is an arbitrary singular variable of any sort, vv is an arbitrary plural variable of any sort, t stands for any term of the appropriate sort, and $\phi, \psi \in \mathcal{L}^W_{\bar{\mathrm{T}}}$ are arbitrary formulae.*

($\forall$IE) $\forall \boldsymbol{v}\phi \to (\mathrm{E}t \to \phi[t/\boldsymbol{v}])$ *is an axiom.*

($\forall$2E) $\phi \leftrightarrow \forall \boldsymbol{v}\phi$ *is an axiom, provided $\boldsymbol{v}$ is not free in ϕ*

($\forall^{\to}$) $\forall \boldsymbol{v}(\phi \to \psi) \to (\forall \boldsymbol{v}\phi \to \forall \boldsymbol{v}\psi)$ *is an axiom.*

(UG) $\phi \,/\, \forall \boldsymbol{v}\phi$.

(UE) $\forall v \mathrm{E} v$ *is an axiom.*

(UEP) $\forall vv \bar{\mathrm{E}} vv$ *is an axiom.*

As well as (N), (K), (T), (5),[21] (OE), (PC), (MP) and (I1), (I2), (Comp), (NE), (Ext), (P1), (QE), (T[]), (P), (PE), (PNeg1), (PNeg2), and ($\bar{\mathrm{T}}$).

This completes the presentation of the propositional contingentist logic $\vdash_c$.

[21] Axiom (5) plays a key role in argument to follow and yet, as a reviewer noted, some contingentists reject S5 as the correct logic for metaphysical modality, see (Adams, 1981) and (Fitch, 1996). However, the classic Adams-Fitch style worry presupposes serious actualism, which fails to hold in $\vdash_c$. Moreover, even if we assume serious actualism, many have argued that the classic Adams-Fitch style worry about S5 relies on faulty presuppositions about world-relative truth, see (Menzel, 1991: 355–356), (Turner, 2005: 203–207), (Einheuser, 2012: 15–17), and (Mitchell-Yellin and Nelson, 2016).

3 Possible Worlds in Logic $\vdash_c$

I establish the more general result that *no* adequate notion of a possible world can be given in $\vdash_c$, by focusing on uncontroversial elements of worlds—that worlds are consistent and that propositions are true relative to them. That is to say, I do not focus on the particularities of this or that definition of a possible world in $\vdash_c$. Instead, I focus on the unproblematic definitions available to the propositional contingentist for consistency and world-relative truth which require little discussion. I then show that any account of worlds in $\vdash_c$ which takes them to have such features will be inadequate. Since all accounts of worlds should incorporate such features, no notion of worlds in $\vdash_c$ is adequate.

I argue for this general result by, first, defining consistency and the relation of world-relative truth, using the notation introduced earlier. Straightforwardly, we say that propositions pp are *consistent* just in case $\Diamond\overline{\mathrm{T}}pp$, where $\overline{\mathrm{T}}$ is the primitive plural truth predicate outlined earlier. Truth in a world is defined in $\vdash_c$ much in the same way as in $\vdash_n$, i.e., a proposition is true in a world if its truth is necessitated by the joint truth of those propositions which are the world.

Definition 17 ($\models$) *For any proposition p: $\ulcorner pp \models p \urcorner$ abbreviates $\ulcorner \Box(\overline{\mathrm{T}}pp \rightarrow \mathrm{T}p) \urcorner$.*

Second, I let $\mathrm{W}^{\mathcal{C}}$ be a schematic term which stands for *any* proposed world-hood predicate. Since any viable account of possible worlds in $\vdash_c$ must take them to be at least consistent, contingentists can only endorse some world-hood predicate $\mathrm{W}^{\mathcal{C}}$ for which the following is satisfied.

$$\text{(W) } \mathrm{W}^{\mathcal{C}}pp \rightarrow \Diamond\overline{\mathrm{T}}pp$$

To spell this out a little further, suppose the contingentist proposes some world-hood predicate $\mathrm{W}^{\mathcal{C}_1}$ in $\vdash_c$. Needless to say, a discerning factor in whether some pp satisfy $\mathrm{W}^{\mathcal{C}_1}$ is whether pp are consistent. In other words, they should endorse the $\mathrm{W}^{\mathcal{C}_1}$-instance of (W). To be clear, the requirement here is not that the contingentist *defines* their world-hood predicate in simpler terms—it may be taken as primitive. The claim is simply that for contingentist theory to be adequate, the appropriate instance of (W) must be true. Using (W), we can make a more general argument, by focusing on all and only those potential world-hood predicates which meet this minimal condition.

Crucially, we can show that, although minimal, if the contingentist articulates an account of worlds in $\vdash_c$ which satisfies (W), then one direction of each of the fundamental theorems holds in $\vdash_c$. However, we can also show that no contingentist theory of worlds can guarantee that *both* directions of each of the fundamental theorems are true, if we assume, as we should, that (W) is satisfied. First, the positive results. Note that, given (T[]), a corresponding version of L.10 holds in $\vdash_c$:

Lemma 18 *For any formula* $\phi \in \mathcal{L}^{W}_{\overline{\mathrm{T}}}$: $\vdash_c \Diamond \mathrm{T}[\phi] \leftrightarrow \Diamond\phi$.

Thus, for any contingentist theory of worlds which satisfies (W), one direction of each of the Fundamental Theorems holds in $\vdash_c$, just as in $\vdash_n$.

Theorem 19: $\vdash_c \exists pp(\mathrm{W}^{\mathcal{C}} pp \wedge pp \models [\phi]) \rightarrow \Diamond\phi$, *for any formula* $\phi \in \mathcal{L}^{W}_{\overline{\mathrm{T}}}$ *with no free occurrence of 'pp', assuming that* $\mathrm{W}^{\mathcal{C}}$ *satisfies* (W).

Proof. By deductions in $\vdash_c$.

(1) $\vdash_c \exists pp(\mathrm{W}^{\mathcal{C}} pp \wedge pp \models [\phi]) \rightarrow \exists pp(\Diamond \overline{\mathrm{T}} pp \wedge \Box(\overline{\mathrm{T}} pp \rightarrow \mathrm{T}[\phi]))$ (W), D.17

(2) $\vdash_c \exists pp(\Diamond \overline{\mathrm{T}} pp \wedge \Box(\overline{\mathrm{T}} pp \rightarrow \mathrm{T}[\phi])) \rightarrow \Diamond \mathrm{T}[\phi]$ QML

(3) $\vdash_c \exists pp(\mathrm{W}^{\mathcal{C}} pp \wedge pp \models [\phi]) \rightarrow \Diamond\phi$ (1), (2), L.18

$\Box$

Theorem 20: $\vdash_c \Box\phi \rightarrow \forall pp(\mathrm{W}^{\mathcal{C}} pp \rightarrow pp \models [\phi])$, *for any formula* $\phi \in \mathcal{L}^{W}_{\overline{\mathrm{T}}}$ *with no free occurrence of 'pp', where* $\mathrm{W}^{\mathcal{C}}$ *satisfies* (W).

Proof. By deductions in $\vdash_c$.

(1) $\vdash_c \Box\phi \rightarrow \Box(\overline{\mathrm{T}} pp \rightarrow \mathrm{T}[\phi])$ (T[]), (N), (K)

(2) $\vdash_c \Box\phi \rightarrow (\mathrm{W}^{\mathcal{C}} pp \rightarrow \Box(\overline{\mathrm{T}} pp \rightarrow \mathrm{T}[\phi]))$ (1), (PC), (MP)

(3) $\vdash_c \forall pp(\Box\phi \rightarrow (\mathrm{W}^{\mathcal{C}} pp \rightarrow \Box(\overline{\mathrm{T}} pp \rightarrow \mathrm{T}[\phi])))$ (2), (UGP)

(4) $\vdash_c \forall pp \Box \phi \to \forall pp(\mathrm{W}^{\mathcal{C}} pp \to \Box(\bar{\mathrm{T}} pp \to \mathrm{T}[\phi]))$ (3), $(\forall^{\to})$

(5) $\vdash_c \Box \phi \to \forall pp(\mathrm{W}^{\mathcal{C}} pp \to \Box(\bar{\mathrm{T}} pp \to \mathrm{T}[\phi]))$ (4), $\forall$2E

□

The other directions of the theorems are more demanding, however, since they require that the space of possible worlds is, in a sense, complete. In fact, I will now show that the other direction of the fundamental theorem for possibility *cannot* be taken as a general and necessary truth by the contingentist. This follows from some important results concerning *consistent pluralities of propositions* and claims about *possibly indistinguishable entities* in $\vdash_c$.

First, let's start with the intuitive shape of the problem for the contingentist. Very generally, we say that two objects are indistinguishable in some respect, if there are no propositions available which we can use to distinguish those two objects in that respect. Here, we 'use propositions' to distinguish objects by supposing they are true, and seeing what follows. In other words, then, two objects x and y are indistinguishable with respect to, say, ϕ, if there are no propositions the truth of which necessitate that $\phi(x)$, but not that $\phi(y)$, or *vice versa*. Of course, it is controversial whether there even possibly are objects which are indistinguishable in *every* respect, i.e., for any open-sentence $\phi \in \mathcal{L}^{W}_{\mathrm{T}}$. However, for our present purposes, we need only be concerned with the idea of objects being indistinguishable with respect to *existence* and, for the rest of the paper, I will refer to this simply as 'indistinguishability'. Formally, this can be defined:

Definition 21 *$\ulcorner x \approx_e y \urcorner$ abbreviates $\ulcorner \forall pp(\Box(\bar{\mathrm{T}} pp \to \mathrm{E}x) \leftrightarrow \Box(\bar{\mathrm{T}} pp \to \mathrm{E}y)) \urcorner$.*

That is to say, x and y are indistinguishable just in case, for any plurality of propositions, *pp*, necessarily, if *pp* are true, then x exists *if and only if*, necessarily, if *pp* are true, then y exists.

Now, the problem for the contingentist is that the existence of adequate possible worlds is inconsistent with certain well-motivated claims about *distinct* and yet *possibly indistinguishable* entities in the above sense. Loosely speaking, if the systematic connection between possibility and truth in some world holds, then all possibility claims are 'witnessed', as it were, by some propositions—if some claim is possible, its truth is necessitated by some propositions which qualify as a world. However, as I will argue, there are well-motivated claims for the propositional contingentist about possibilities involving entities which are possibly indistinguishable in the above sense. Thus, I will argue that there are claims which the contingentist must accept as possible and yet the truth of those claims are not necessitated by any propositions—*a fortiori* those claims are not necessitated by any propositions which qualify as a world. Since the propositional contingentist *should* accept such claims about genuinely distinct but possibly indistinguishable entities, the failure of the systematic connection between possibility and truth in some world *follows*. (Later, I forestall worries about this argument and show that there are models of a sound semantics for $\vdash_c$ in which such well-motivated indistinguishability claims are true.)

Here's how we make this problem precise in $\vdash_c$. First, consider this result in $\vdash_c$.

Lemma 22 $\vdash_c x \approx_e y \rightarrow \neg\exists pp(\Diamond\overline{\mathrm{T}}pp \wedge (pp \models [\mathrm{E}x \wedge \neg\mathrm{E}y]))$.

Proof. By deductions in $\vdash_c$. *For convenience, let* $\phi^{x,y} := \mathrm{E}x \wedge \neg\mathrm{E}y$

(1) $\exists pp(\Diamond\overline{\mathrm{T}}pp \wedge pp \models [\phi^{x,y}]) \rightarrow \exists pp(\Diamond\overline{\mathrm{T}}pp \wedge \Box(\overline{\mathrm{T}}pp \rightarrow \mathrm{T}[\phi^{x,y}]))$ D.17

(2) $\Box(\mathrm{T}[\phi^{x,y}] \rightarrow \mathrm{E}x)$ (T[]), (PC), (N)

(3) $\Box(\mathrm{T}[\phi^{x,y}] \rightarrow \neg\mathrm{E}y)$ (T[]), (PC), (N)

(4) $\exists pp(\Diamond\overline{\mathrm{T}}pp \wedge pp \models [\phi^{x,y}]) \rightarrow \exists pp(\Box(\overline{\mathrm{T}}pp \rightarrow \mathrm{E}x) \wedge \Box(\overline{\mathrm{T}}pp \rightarrow \neg\mathrm{E}y))$ (1), (2), (3)

(5) $\exists pp(\Box(\overline{\mathrm{T}}pp \rightarrow \mathrm{E}x) \wedge \Box(\overline{\mathrm{T}}pp \rightarrow \neg\mathrm{E}y)) \rightarrow \neg x \approx_e y$ D.21, QML

(6) $\exists pp(\Diamond\overline{\mathrm{T}}pp \wedge pp \models [\phi^{x,y}]) \rightarrow \neg x \approx_e y$ (4), (5), (T[]), (N), QML

Contrapose (6) *for the result.* □

(22) shows that if x and y are indistinguishable in the sense of (21), then there are no possibly true propositions, the joint truth of which necessitate the truth of the proposition that x exists but y does not. As we saw earlier, any adequate notion of a world will satisfy (W), i.e., the propositions which are the world will be at least jointly possibly true. Thus, it is easy to see that (22) shows that if x and y are indistinguishable, then there are no *worlds* relative to which the proposition that x exists but y does not is true. Note that (22) is not problematic

for the propositional contingentist in and of itself. Rather, the problem arises due to the interaction in $\vdash_c$ between (22) and some well-motivated claims for the propositional contingentist about distinct and yet possibly indistinguishable entities. Here, I will focus explicitly on the most plausible and minimal such claim and the argument against contingentist worlds we can develop from this claim.

Consider, then, the following.

(ID) $\Diamond\exists x\Diamond\exists y(\Diamond x \approx_e y \wedge \Diamond(\mathrm{E}x \wedge \neg\mathrm{E}y))$
(To be read: Possibly there is something x, such that possibly there is something y and x and y are possibly indistinguishable and possibly x exists and y does not.)

(ID) does not involve claiming that there *are* objects which could have been indistinguishable. Rather, it is the claim that it is possible that there is *some* object such that it is possible for there to be *another* which is possibly indistinguishable from the first.[22] (ID) is a logically weak, and plausible, claim indeed. To see why the propositional contingentist ought to accept it, it's worth considering what philosophical picture emerges if one rejects (ID) and how implausible such a metaphysics is. According to such a metaphysics, at least the following holds.

($\neg$ID) $\Box\forall x\Box\forall y(\Diamond(\mathrm{E}x \wedge \neg\mathrm{E}y) \rightarrow \Box\neg x \approx_e y)$
(To be read: Necessarily, for any thing x and necessarily for any thing y, if it is possible that x, but not y, exists, then necessarily x and y are distinguishable.)

That is to say, speaking loosely, for any two individuals x and y across all possible worlds, if it is possible for either one to exist without the other, then in every world w, those x and y are distinguishable in terms of propositions which exist in w. The obvious test case for whether ($\neg$ID) is acceptable to the propositional contingentist is the case where x and y can both fail to exist. To accept ($\neg$ID) is to think that, even if both x and y were not to exist, there would be propositional resources available to distinguish x from y. Of course, it's clear that the necessitist can accept this. In each world, there will be propositions which distinguish x from y and *vice versa*, e.g., propositions like $[\mathrm{E}x]$ or $[\mathrm{E}y]$. Likewise, a contingentist who only accepts that certain individuals, but not propositions, contingently exist could accept ($\neg$ID).

[22] I should be clear: the claim here is not that (ID) is a theorem of $\vdash_c$. It is not one of the general and necessary truths distinctive of the propositional contingentist view. Rather, it is a matter of particular fact which the contingentist has good reason to suppose holds.

However, it is clear that for the *propositional* contingentist no propositions like $[\mathrm{E}x]$ or $[\mathrm{E}y]$ would exist, if x and y were not to exist. The propositional contingentist can accept (¬ID) only if they think that, even in such cases as a contingently existing x and y, there are some qualitative propositions which necessarily uniquely specify x and y, i.e., propositions not expressed by sentences featuring 'x' or 'y'. This, however, implies a deeply implausible metaphysical picture. It is one in which, necessarily, for any two entities which satisfy the antecedent of (¬ID), there is some qualitative feature of either x and y which we can use to lock onto x or y independently of x or y existing. For instance, if x and y were two qualitatively indistinguishable electrons, it is deeply implausible that there *necessarily* exist propositions which we can use to distinguish x from y.[23]

One natural thought is that one may endorse (¬ID), if one appealed to special kinds of properties like essences or thisnesses. For instance, Plantinga has defended the existence of a special kind of property, known as an essence, which, for each individual, uniquely tracks that individual across all worlds (Plantinga, 1979). An essence for some individual is exemplified only by that individual, in any world that individual exists. Importantly, such essences are qualitative and exist necessarily, regardless of whether the individual itself exists. Likewise, some have argued that there are non-qualitative properties known as thisnesses which similarly uniquely track individuals across worlds and which can exist independently of the existence of those individuals, see (Ingram, 2018b) and (Ingram, 2018a). However, it is implausible that such properties are a well-motivated addition to the propositional contingentist's ontology. In the case of qualitative essences, it is deeply implausible that, and under-explained how, for every individual, there is a qualitative property which is able to lock on, and uniquely track, an individual across all possible worlds, see (Williamson, 2013: 269). In the case of thisnesses, the situation is even less motivated. Given the close parallel between propositions and properties, it is deeply problematic for the propositional contingentist to take certain *propositions* like [John exists] to ontologically depend on the objects they concern, and yet deny that non-qualitative *properties* defined in terms of, or built out of, certain individuals ontologically depend on those very individuals. (See (Fine, 1985: 149) and (Williamson, 2013: Chp. 6) for a discussion of this, and similar, issues.)

For the propositional contingentist, then, the metaphysical burden of denying (ID) is simply

[23] Compare this example to similar cases discussed in (Williamson, 2013: 272–274), (Fritz and Goodman, 2016: 649–650), and (Stalnaker, 2012: 18–19).

an anathema to their view. Propositional resources which uniquely identify even a contingent individual across all possible worlds are precisely those resources which the propositional contingentist rejects. Thus, the propositional contingentist should accept (ID)—it is possible for there to be an object for which it is possible that there is another and the two are possibly indistinguishable and yet, because they are distinct objects, it is possible for one to exist and yet the other not. The problem, in the present context, is that the following results show that accepting a well-motivated claim like (ID) means that theoretically adequate possible worlds are not available to the propositional contingentist.

Lemma 23 $\vdash_c \mathrm{ID} \rightarrow \Diamond\exists x\Diamond\exists y(\Diamond(\mathrm{E}x \wedge \neg\mathrm{E}y) \wedge \Diamond\neg\exists pp(\Diamond\overline{\mathrm{T}}pp \wedge pp \models [\mathrm{E}x \wedge \neg\mathrm{E}y]))$

Proof. The following is derived as a corollary from (22) *with* (N) *and* UG*:*

$$\text{(i)} \vdash_c \Box\forall x\Box\forall y(x \approx_e y \rightarrow \neg\exists pp(\Diamond\overline{\mathrm{T}}pp \wedge pp \models [\mathrm{E}x \wedge \neg\mathrm{E}y]))$$

From some simple QML *and* (i), (23) *follows.* □

Crucially, there is no incoherence in $\vdash_c$ in the supposition that (ID) is true. That is to say, there are models in which all the axioms of $\vdash_c$ are true and the inference rules truth-preserving and in which (ID) holds. In the appendix, I outline such a semantics, showing that $\vdash_c$ is sound with respect to a class of models of that semantics, $\mathbb{M}^P$. Here I simply state the relevant result.

Theorem 24 *For some* $\mathcal{M} \in \mathbb{M}^P$*:* $\mathcal{M}, w, \underline{a} \vDash_c \mathrm{ID}$

(23) and (24) have the following consequence.

Theorem 25 *It is not the case that, for any formula* $\phi \in \mathcal{L}^W_{\mathrm{T}}$ *with no free* pp: $\vDash_c \Diamond\phi \rightarrow \exists pp(\mathrm{W}^{\mathcal{C}}pp \wedge pp \models [\phi])$*, assuming that* $\mathrm{W}^{\mathcal{C}}$ *satisfies* (W)

Proof. Let:

$$\xi := \Diamond\exists x\Diamond\exists y(\Diamond(\mathrm{E}x \wedge \neg\mathrm{E}y) \wedge \Diamond\neg\exists pp(\Diamond\overline{\mathrm{T}}pp \wedge pp \models [\mathrm{E}x \wedge \neg\mathrm{E}y]))$$
$$\psi := \neg\Box\forall x\Box\forall y(\Diamond(\mathrm{E}x \wedge \neg\mathrm{E}y) \rightarrow \Box\exists pp(\mathrm{W}^{\mathcal{C}}pp \wedge pp \models [\mathrm{E}x \wedge \neg\mathrm{E}y])))$$

Assume $\mathrm{W}^{\mathcal{C}}$ *satisfies* (W), *then:* (i) $\vdash_c \xi \rightarrow \psi$ *and thus, given the soundness of the semantics based on* $\mathbb{M}^P$, (ii) $\vDash_c \xi \rightarrow \psi$. *Next, suppose that* $\vDash_c \Diamond\phi \rightarrow \exists pp(\mathrm{W}^{\mathcal{C}}pp \wedge pp \models [\phi])$, *for any* $\phi \in \mathcal{L}^W_{\mathrm{T}}$ *with no free* pp. *Letting* $\phi := \mathrm{E}x \wedge \neg\mathrm{E}y$, *it follows, by soundness, the supposition,* S5, *and applications of* (UG) *and* (N) *that* $\vDash_c \neg\psi$. *However, by* (23), (24), (ii), *and soundness, we know that there is an* $\mathcal{M} \in \mathbb{M}^P$ *such that* $\mathcal{M}, w, \underline{a} \vDash_c \xi$ *and thus* $\mathcal{M}, w, \underline{a} \vDash_c \psi$. *Thus, it is not the case that* $\vDash_c \Diamond\phi \rightarrow \exists pp(\mathrm{W}^{\mathcal{C}}pp \wedge pp \models [\phi])$, *for any* $\phi \in \mathcal{L}^W_{\mathrm{T}}$ *with no free* pp, *assuming* $\mathrm{W}^{\mathcal{C}}$ *satisfies* (W). □

Theorem 26 $\nvdash_c \Diamond\phi \rightarrow \exists pp(\mathrm{W}^{\mathcal{C}}pp \wedge pp \models [\phi])$, *for any* $\phi \in \mathcal{L}^W$ *with no free* pp, *assuming that* $\mathrm{W}^{\mathcal{C}}$ *is some world-hood predicate which satisfies* (W).

Proof. Corollary of (25) *and soundness* (*see appendix*) □

To summarise, (ID) is a particularly logically weak and plausible claim and the propositional contingentist ought to accept it. However, if it holds, then the fundamental theorem for possibility is not provable in $\vdash_c$, for any conception of worlds which takes them to be consistent. Crucially, it is not an option for the propositional contingentist to simply supplement $\vdash_c$ with the fundamental theorems as axioms. The presence of the systematic connection between possibility and truth at some world amongst the general and necessary truths of the propositional contingentist's view is *inconsistent* with the claim that there might have existed distinct and yet possibly indistinguishable entities. If the fundamental theorems were stipulated to hold, the propositional contingentist would have to accept (¬ID) as a theorem—as a general and necessary truth of their view. Such a consequence is, however, deeply unpalatable, as I have already argued.

In the next section, I will outline a further limitative result for the contingentist. But before doing so, it's worth discussing the generality of the argument presented so far. Explicitly, the arguments presented here concern the prospects of the contingentist securing the existence of theoretically adequate worlds, where worlds are pluralities of propositions. However, in several important respects, the arguments presented are plausibly more general. First, I have not assumed that the contingentist *defines* worlds as those pluralities of propositions which are at least possibly jointly true. Rather, I plausibly assume that, regardless of the constraints the contingentist imposes on pluralities of propositions, it will be nonetheless true that every such plurality is, as a matter of fact, at least possibly jointly true. Similarly, though the definition of *truth in a world* is modal, provided a proposition p is true in a world *pp only if* the truth of p is necessitated by *pp*, regardless of how this notion is strictly defined, the arguments above apply: the arguments show that the possibility of indistinguishable entities rules out the existence of pluralities *pp* and propositions p which meet minimal *necessary* conditions for being a world and for being a proposition true in such a world.

Moreover, the arguments here plausibly generalise to other potential conceptions of contingentist possible worlds, given the close parallels between pluralities of propositions and other entities like individual propositions, sets of propositions, states of affairs, or properties. Here, we define the 'actuality', and the consistency or possibility of a world using the notion of singular, and plural, propositional *truth*. If worlds, however, were, say, properties, these notions wouldn't apply. However, analogous ones would, e.g., a world w would be actual if *instantiated* and p would be true in w, if the truth of p were necessitated by the instantiation of w. Now, plausibly, the propositional contingentist is committed to the same patterns of contingency in what properties there are as with propositions (Williamson, 2013: 289), and indeed, are plausibly committed to the same patterns of contingency in what sets of propositions, or states of affairs, there are. Thus, the thought goes, the arguments presented here should be of general concern to conceptions of contingentist possible worlds beyond the specific proposal in which such arguments are presented, given that many such alternative conceptions of worlds will involve positing entities exhibiting deeply analogous features, allowing us to reformulate the specific versions of the arguments presented here. It goes without saying, of course, that the arguments here will not generalise to *all* conceptions of contingentist worlds. For instance, the contingentist could put forward an account of worlds in which the notion of truth in a world is not understood to entail that the relevant proposition's truth is necessitated by the actuality of the world. However, my point here is only that the arguments can be very plausibly extended to many other approaches to worlds which eschew the specific details involved in formulating them here.

It's also worth discussing how the results in this section relate to other results in the extant literature, particularly those in (Fritz, 2016). There, Fritz develops two classes of models to model contingency in what propositions there are.[24] Fritz then raises worries for the prospects

[24]Though Fritz develops two classes of models for propositional contingentism—equivalence, and permutation, systems—they are shown to be equivalent in the sense of admitting the same patterns of contingency (Fritz, 2016:

of understanding world-talk in terms of maximally strong non-trivial propositions (Ibid.: 140–141), as suggested in (Stalnaker, 1976). Now, one may worry that the results presented here are no more than notational variants of results in (Fritz, 2016). This is not, however, the case. First, we are here concerned with a different conception of propositional contingentism. Distinguish:

Aboutness View Some propositions p are *directly*, or *singularly* about individuals $i_1, ..., i_n$ and because of this relation between p and $i_1, ..., i_n$, if any of $i_1, ..., i_n$ were not to exist, p would not.

Distinction View A proposition p is in part the proposition that it is because of the distinctions it draws in modal space. Some propositions draw distinctions which essentially involve appealing to individuals $i_1, ..., i_n$. In the absence of $i_1, ..., i_n$, such propositions do not exist.[25]

Fritz is explicitly interested in developing models for propositional contingentism according to the *distinction* conception, see (Fritz, 2016: 124; 2018a: 408) and see also (Fritz and Goodman, 2016: 646; 2017a: 509). The present paper has been concerned with an aboutness view: propositions are contingent entities if they are expressed by sentences which feature terms denoting contingent entities. These two ways of motivating propositional contingentism are fundamentally different and lead to diverging judgements on important cases. For instance, consider a tautologous proposition like $[Fq \vee \neg Fq]$. Such a proposition, according to the aboutness view is a contingent entity, if q is contingent. However, according to the distinction view, $[Fq \vee \neg Fq]$ is a necessarily existing proposition: the trivial distinction it draws in modal space does not essentially involve appealing to the individual q (Stalnaker, 2012). One cannot assume that results about one conception of propositional contingentism simply carry over to the other.

More importantly, the results presented here are independent to those in (Fritz, 2016) and, in a sense, more flexible. Fritz is concerned with one plausible way the propositional contingentist could understand world-talk. As I've stressed, the results presented here make few assumptions about worlds and generalise to many conceptions of worlds which make the same, or analogous, assumptions. The arguments here also show that such a minimal assumptions

131). Here is not the place to outline Fritz's work in detail, see (Fritz, 2016). Thanks to a reviewer for emphasising the need to contrast my result's and Fritz's.

[25]The distinction view is explored and defended in print by Fine (1977, 1980), and Stalnaker (2012). The more common aboutness view is defended and discussed most prominently in (Prior, 1957), (Adams, 1981), (Deutsch, 1990), (Menzel, 1991), (Fitch, 1996), (Turner, 2005), and (Einheuser, 2012).

entail that the contingentist should reject even very weak claims like (ID). This contrasts with the discussion in (Fritz, 2016). His primary concern there is how one particular strategy for understanding world-talk interacts with generalised quantifier phrases like 'there are uncountably many worlds such that...', especially given the limitative results about such quantifiers in higher-order modal languages obtained in (Fritz, 2018b). Thus, even if it were legitimate to generalise results in (Fritz, 2016) about the distinction view to the aboutness view, my results show that, for the contingentist, it is not enough to respond to Fritz's worries, since there are independent and stubborn worries about contingentist possible worlds.

4 Actualised Fundamental Theorem for Possibility

I want to finally discuss one further limitative result since it shines a light on a tempting, but mistaken, response which the propositional contingentist may wish to take. One may naturally worry that what has driven the arguments here is the requirement that the systematic connections between possibility, necessity, and truth at possible worlds hold of *unqualified necessity*. Instead, one might think that what is important is that such connections hold *here*, in the *actual* world. After all, the results presented so far simply show that the fundamental theorems *possibly fail to be true*, if some claims about distinct but merely possible individuals are true. Instead, one might think that the machinery of possible worlds can help elucidate modality even if it just gives us purchase on those modal claims which are actually true. The objection would then run: what needs to be shown, and crucially what hasn't been shown, is that the fundamental theorems fail to *actually hold* and without this result my argument is in jeopardy. For the rest of the paper, then, I will argue for just that—*even if* we only require the fundamental theorem to actually hold, we can outline a similar limitative argument against the contingentist.

To better understand this response, we should first spend some time talking about how an actuality operator should work in some suitable extension of $\vdash_c$. If we extend $\mathcal{L}^W_{\mathrm{T}}$ to include an operator @ for actuality, we will need to supplement (16) to include some axioms or rules to govern it as well as extending the notion of a formula of $\mathcal{L}^W_{\mathrm{T}}$. Both tasks are straightforward. First, we say that if formula $\phi \in \mathcal{L}^W_{\mathrm{T}@}$, then formula $@\phi \in \mathcal{L}^W_{\mathrm{T}@}$, where $\mathcal{L}^W_{\mathrm{T}@}$ is $\mathcal{L}^W_{\mathrm{T}}$ extended to include @. Second, we supplement $\vdash_c$ with the following principles for @ to get a proof system $\vdash^{@}_c$:

(@1) ϕ / $@\phi$.

(@2) $@(\phi \to \psi)$ / $@\phi \to @\psi$

(@3) $\vdash^{@}_{c} \neg @\phi \leftrightarrow @\neg\phi$

(@4) $\vdash^{@}_{c} @\phi \to \Box @\phi$

Note that these principles are not intended to be a full axiomatization of @. (@1)–(@4) are the least which hold of '@'. Now, weakening the requirement to only the *actualised* fundamental theorem means that the propositional contingentist takes the following to be a validation of the adequacy of some notion of worlds $\mathrm{W}^{\mathcal{C}}$, for any formula $\phi \in \mathcal{L}^{W}_{\mathrm{T}_{@}}$:

$$\vdash^{@}_{c} @(\Diamond\phi \leftrightarrow \exists pp(\mathrm{W}^{\mathcal{C}} pp \wedge pp \models [\phi]))$$

This becomes the requirement that, for any formula $\phi \in \mathcal{L}^{W}_{\mathrm{T}_{@}}$:

(@) $\vdash^{@}_{c} \Diamond\phi \leftrightarrow @\exists pp(\mathrm{W}^{\mathcal{C}} pp \wedge pp \models [\phi])$

This follows straightforwardly from (@1)–(@4) and, given that the underlying modal logic is S5, the fact that $\vdash^{@}_{c} \Diamond\phi \leftrightarrow @\Diamond\phi$.

Thus the question of whether the propositional contingentist can endorse the mere actuality of the fundamental theorems becomes the question of whether the propositional contingentist can guarantee that (@) holds.[26] The following argument shows that they cannot meet this requirement, since the same sort of problem reemerges in this setting. First, consider the following.

(ID@) $\Diamond\exists x\exists y(@x \approx_e y \wedge \Diamond(\mathrm{E}x \wedge \neg\mathrm{E}y))$
(To be read: Possibly there exist two things x and y and x and y are actually indistinguishable and it is possible that x exists and y does not.)

Of course, insofar as we cannot distinguish x and y in terms of actually existing propositions, x and y are entities which do not actually exist. However, crucially, just as before, in endorsing (ID@) we do not talk about two merely possible and indistinguishable individuals *in particular* but only two merely possible and indistinguishable individuals *whichever* two things in particular they would turn out to be. The problem for the propositional contingentist is that (ID@) is

[26] When discussing modal logics featuring operators like @, often a distinction is made between *general validity*—truth at every world in every model—and *real world validity*—truth at the distinguished "actual" world in every model (Davies and Humberstone, 1980: 1). This may prompt the thought that what the propositional contingentist may simply want is that the fundamental theorems come about as *real world* logical truths. However, a corollary of the limitative result to follow is that the unactualised fundamental theorem for possibility is not real world valid—see Theorem 50.

just as problematic as (ID).

Lemma 27 $\vdash^{@}_{c}$ ID@ $\rightarrow \Diamond\exists x\exists y(\Diamond(\mathrm{E}x \wedge \neg\mathrm{E}y) \wedge \neg @\exists pp(\Diamond\overline{\mathrm{T}}(pp) \wedge pp \models [\mathrm{E}x \wedge \neg\mathrm{E}y]))$

Proof. Again, from (22), *we know that*

$$\vdash^{@}_{c} x \approx_{e} y \rightarrow \neg\exists pp(\Diamond\overline{\mathrm{T}}pp \wedge pp \models [\mathrm{E}x \wedge \neg\mathrm{E}y])$$

From this we derive the result, using (@1)-(@3) *and* (UG), (N), *and* (W) □

Again, we appeal to some model-theoretic results to show that there is no incoherence in the supposition that there might have been some individuals which are actually indistinguishable and which might have existed without the other existing. Only this time, we appeal to a different class of models, $\mathbb{M}^{@}$. I show that $\vdash^{@}_{c}$ is sound with respect to $\mathbb{M}^{@}$ in the appendix. Here is the relevant result, where $\models^{@}_{c}$ is the notion of truth in a $\mathbb{M}^{@}$ model.

Theorem 28 *For some* $\mathcal{M} \in \mathbb{M}^{@}$*:* $\mathcal{M}, w, \underline{a} \models^{@}_{c}$ ID@

(27) and (28) have much the same effect as the previous results. If (@) holds, then so too does the following.

(@*) $\vdash^{@}_{c} \Box\forall x\forall y(\Diamond(\mathrm{E}x \wedge \neg\mathrm{E}y) \rightarrow @\exists pp(\Diamond\overline{\mathrm{T}}pp \wedge pp \models [\mathrm{E}x \wedge \neg\mathrm{E}y])$

However, by (28) we know that there is some model $\mathcal{M} \in \mathbb{M}^{@}$ in which ID@ is true at some world, under an assignment. Thus, given (27) and the fact that the semantics defined over $\mathbb{M}^{@}$ is sound for $\vdash^{@}_{c}$, it follows that:

Theorem 29 *It is not the case that* $\models^{@}_{c} \Diamond\phi \rightarrow @\exists pp(\mathrm{W}^{\mathcal{C}}pp \wedge pp \models [\phi])$*, for any formula* $\phi \in \mathcal{L}^{W}_{\overline{\mathrm{T}}@}$ *with no free pp, assuming that* $\mathrm{W}^{\mathcal{C}}$ *satisfies* (W).

Proof. Corollary of (27)–(28) *and* (42) (*see Appendix*). □

Theorem 30 *It is not the case that* $\vdash^{@}_{c} \Diamond\phi \rightarrow @\exists pp(\mathrm{W}^{\mathcal{C}}pp \wedge pp \models [\phi])$*, for any formula* $\phi \in \mathcal{L}^{W}_{\overline{\mathrm{T}}@}$ *with no free pp, assuming satisfies (W).*

Proof. Corollary of (29) *and* (42) (*see Appendix*). □

Thus, even if the contingentist only demands that possible worlds are adequate if the fundamental theorem for possibility *actually* holds, they are unable to define theoretically adequate worlds. The requirement that the fundamental theorem of possibility holds of unqualified necessity and not merely actually is not an essential element of the argument I have presented here.

5 Concluding Remarks

I have presented in detail the two most promising limitative results which show that the propositional contingentist is unable to embed theoretically adequate notions of possible worlds into a natural and promising logic which characterises their view. First, I presented an argument which focused on a compelling claim about distinct and yet indistinguishable entities, showing that there is an inconsistency between such a claim and the availability of adequate possible worlds in $\vdash_c$. I also appealed to the existence of a model for such a claim in a sound semantics for the logics which characterise their view—the existence of such a model is proven in the appendix. Consequently, the fundamental theorem for possibility cannot be a theorem of $\vdash_c$. Given the role of $\vdash_c$, this shows that the systematic connection between possibility and truth at some world is not one of the general and necessary truths distinctive of their view. As a second argument, I considered the question of whether the systematic connection between possibility and truth at some world could be taken to be a general, but only *actual*, truth of the propositional contingentist view. I showed that a parallel argument can be given against the availability of contingentist possible worlds, even if the requirement is weakened in this way. Each of these results contrasts with the situation for the necessitist.

At the very least then, the propositional contingentist cannot utilise the notion of a possible world unreflectively. There are three options for such contingentists. They may wish to theorize about modality without recourse to possible worlds; they may wish to abandon their propositional contingentism in favour of propositional necessitism, a view which can unproblematically theorise with worlds; or they may attempt to circumvent my argument and articulate a theoretically useful notion of a possible world. There are options available to the contingentist who wishes to take the final option. My argument here has involved several choices in formulating contingentism which one could attempt to challenge—perhaps most controversially, I rejected serious actualism. However, what is clear is that the contingentist must first do substantial theoretical work to establish that a useful notion of a possible world is available—the results presented here show that even assuming that this latter option is an available one is problematic for the propositional contingentist, since it very plausibly at least requires rejecting claims about possibly indistinguishable entities—claims they ought to accept.

6 Technical Appendix

Here, I present both a sound semantics for $\vdash_c$ and $\vdash_c^@$, I prove that both $\vdash_c$ and $\vdash_c^@$ are consistent, and prove (24) and (28). The semantics presented involves extending the standard model-theoretic semantics for first-order modal logic with variable domains to include machinery for handling the plural fragment of $\mathcal{L}^W_{\overline{\mathrm{T}}}$, propositions and the term-forming brackets.

First, we define a frame.

Definition 31 (Frame): *Let a* frame *be a triple $\langle W, D_i, D_p \rangle$ where W is a non-empty set, D_i is some function which maps each $w \in W$ to a set $D_i(w)$, and D_p is some function which maps each $w \in W$ to a set $D_p(w) \subseteq \mathcal{P}(W) \times \mathcal{P}(W)$.*

Intuitively, W is a non-empty set of possible worlds, $D_i(w)$ is the set of non-propositional individuals which exist at w and $D_p(w)$ is the set of propositions which exist at w. Following (Fine, 1980), we identify propositions with pairs of sets of worlds, i.e. $\langle \alpha, \beta \rangle \in \mathcal{P}(W) \times \mathcal{P}(W)$. Each pair $\langle \alpha, \beta \rangle$ represents a proposition which exists in each world in β and is true in each world in α. Earlier (§2), I noted that (P) is consistent with conceptions of propositions which individuate propositions *at least* by their truth, and existence, conditions. Here, we only want to develop a sound semantics and, to keep things simple, we select the simplest conception of propositions to build into our model theory in order to validate (P): propositions are individuated by their truth, and existence, conditions *and nothing else*. Moreover, there should always be a background worry that more fine-grained views are incoherent, via Russell-Myhill style arguments, e.g., (Russell, 1903) and (Myhill, 1958).

A model based on frame is defined as follows, where D_p^A is $\mathcal{P}(W) \times \mathcal{P}(W)$ and D_i^A is

$\bigcup_{x \in W} D_i(x)$.[27]

Definition 32 (Model): *Let a model be a quadruple $\langle W, D_i, D_p, v\rangle$, where W, D_i, and D_p constitute a frame and v is the valuation function*:

(i) *v maps each $w \in W$ and non-logical n-place predicate of $\mathcal{L}^W_{\mathrm{T}}$ to some set of n-tuples of $d \in D^A_i \cup D^A_p$.*

(ii) *v maps each $w \in W$ and non-logical n-place plural predicate of $\mathcal{L}^W_{\mathrm{T}}$ to some set of n-tuples of subsets of $D^A_i \cup D^A_p$.*

We evaluate formula $\phi \in \mathcal{L}^W_{\mathrm{T}}$ in models relative to assignments. Truth in a model $\vDash_c$ and denotation relative to an assignment $\delta_{\underline{a}}$ are defined in tandem.

Definition 33 (Denotation and $\vDash_c$): *Let an assignment be a function which maps each individual singular variable to some $d \in D^A_i$, each propositional singular variable to some $d \in D^A_p$, each individual plural variable to some non-empty $d_s \subseteq D^A_i$ and each propositional plural variable to some non-empty $d_s \subseteq D^A_p$. Let the variant assignment $\underline{a}[v/d]$ be the assignment which assigns the same values to the same terms as $\underline{a}$, except assigns d to v. We specify the* truth-set *of ϕ relative to $\underline{a}$ as:*

$$ts_{\underline{a}}(\phi) = \{w \in W : \mathcal{M}, w, \underline{a} \models \phi\}$$

We specify the existence-set *of ϕ relative to $\underline{a}$:*

$$es_{\underline{a}}(\phi^{t_1,\ldots,t_n}) = \{w \in W : \mathcal{M}, w, \underline{a} \models \mathrm{E}/\bar{\mathrm{E}}t_1 \wedge \ldots \wedge \mathrm{E}/\bar{\mathrm{E}}t_n\}.$$

Finally let $v(R)_w$ be the extension of predicate R at w. With these in mind:

(i) *The denotation function $\delta_{\underline{a}}$, relative to $\underline{a}$, is given:*

  (a) $\delta_{\underline{a}}(t) = \underline{a}(t)$, *where t is a variable.*

  (b) $\delta_{\underline{a}}([\phi]) = \langle ts_{\underline{a}}(\phi), es_{\underline{a}}(\phi)\rangle$.

  (c) $\delta_{\underline{a}}(\sim p) = \langle (W - \alpha), \beta\rangle$, if $\delta_{\underline{a}}(p) = \langle \alpha, \beta\rangle$

(ii) *The relation of $\vDash_c$, relative to $\underline{a}$, is given:*

[27] Note, the contingentism explored here licenses the existence of impossible propositions, e.g., $[Rxy \vee \neg Rxy]$, where $\neg\Diamond(\mathrm{E}x \wedge \mathrm{E}y)$. This is a feature and not a bug of the view, see (Salmon, 1987: 97) and (Fritz, Unpublished); but it does require us to be careful in defining the total propositional *domain* for a model: we cannot take D^A_p to simply be $\bigcup_{x \in W} D_p(x)$. Otherwise, no propositional term could be assigned some $\langle \alpha, \varnothing\rangle$ in the relevant models.

(a) $\mathcal{M}, w, \underline{a} \vDash_c Rt_1, ..., t_n$ *iff* $\langle \delta_{\underline{a}}(t_1), ..., \delta_{\underline{a}}(t_n) \rangle \in v(R)_w$

(b) $\mathcal{M}, w, \underline{a} \vDash_c \overline{R}tt_1, ..., tt_n$ *iff* $\langle \delta_{\underline{a}}(tt_1), ..., \delta_{\underline{a}}(tt_n) \rangle \in v(\overline{R})_w$

(c) $\mathcal{M}, w, \underline{a} \vDash_c \neg\phi$ *iff* $\mathcal{M}, w, \underline{a} \nvDash \phi$[28]

(d) $\mathcal{M}, w, \underline{a} \vDash_c \forall x \phi$ *iff for every* $d \in D_i(w)$*,* $\mathcal{M}, w, \underline{a}[x/d] \vDash \phi$

(e) $\mathcal{M}, w, \underline{a} \vDash_c \forall p \phi$ *iff for every* $d \in D_p(w)$*,* $\mathcal{M}, w, \underline{a}[p/d] \vDash \phi$

(f) $\mathcal{M}, w, \underline{a} \vDash_c \forall xx \phi$ *iff for every non-empty* $d_s \subseteq D_i(w)$*,* $\mathcal{M}, w, \underline{a}[xx/d_s] \vDash \phi$

(g) $\mathcal{M}, w, \underline{a} \vDash_c \forall pp \phi$ *iff for every non-empty* $d_s \subseteq D_p(w)$*,* $\mathcal{M}, w, \underline{a}[pp/d_s] \vDash \phi$[29]

(h) $\mathcal{M}, w, \underline{a} \vDash_c \Box\phi$ *iff for every* $w \in W$ $\mathcal{M}, w, \underline{a} \vDash \phi$

(i) $\mathcal{M}, w, \underline{a} \vDash_c \Diamond\phi$ *iff for some* $w \in W$ $\mathcal{M}, w, \underline{a} \vDash \phi$

(j) $\mathcal{M}, w, \underline{a} \vDash_c t_1 = t_2$ *iff* $\delta_{\underline{a}}(t_1) = \delta_{\underline{a}}(t_2)$

(k) $\mathcal{M}, w, \underline{a} \vDash_c t \prec tt$ *iff* $\delta_{\underline{a}}(t) \in \delta_{\underline{a}}(tt)$

Let $\ulcorner w \rhd \delta_{\underline{a}}(t) \urcorner$ *be* $\ulcorner \delta_{\underline{a}}(t) =$ *some* $\langle \alpha, \beta \rangle$ *such that* $\alpha, \beta \in \mathcal{P}(W)$ *and* $w \in \alpha \urcorner$ *and* $\ulcorner w \rhd \delta_{\underline{a}}(tt) \urcorner$ *be* $\ulcorner \delta_{\underline{a}}(tt) =$ *some* $d_s \subseteq \mathcal{P}(W) \times \mathcal{P}(W)$ *and, for every* $\langle \alpha, \beta \rangle \in d_s$*,* $w \in \alpha \urcorner$:

(l) $\mathcal{M}, w, \underline{a} \vDash_c \mathrm{T}t$ *iff* $w \rhd \delta_{\underline{a}}(t)$

(m) $\mathcal{M}, w, \underline{a} \vDash_c \overline{\mathrm{T}}tt$ *iff* $w \rhd \delta_{\underline{a}}(tt)$

A formula is true *simpliciter* at a world, if it is true under every assignment at that world. It is *valid in a model*, if it is true at any world in that model. It is *valid relative to a frame*, if it is true in any model on that frame. It is *valid simpliciter*, if it is true in any model based on any frame. For soundness, we first show that a truncated proof system $\vdash_c^{-P}$ is sound with respect to the semantics defined by the more general class of models.

Definition 34 ($\vdash_c^{-P}$): *Let* $\vdash_c^{-P}$ *be* $\vdash_c$ *without* (P) *as an axiom.*

Theorem 35 ($\vdash_c^{-P}$ SOUNDNESS): *If* $\vdash_c^{-P} \phi$*, then* $\vDash_c \phi$*, for any formula* $\phi \in \mathcal{L}^W_{\mathrm{T}}$.

Proof. We show that each axiom except (P) *is valid and the inference rules preserve truth in a model.*

[28]The other logical connectives are given the standard semantic clauses.
[29]Existential quantification is treated as the dual of the respective clauses for universal quantification.

It is immediate that the modal axioms, (PC)*, the axioms for both singular and plural quantification are valid, and the rules for quantification and* (MP) *preserve truth. What remains are the axioms for identity, pluralities, and propositions. In the case where the axiom applies to both propositional and non-propositional sorts, I explicitly prove only the non-propositional version, leaving the second to be established by symmetry with the first.*

IDENTITY. (I1) *is valid since trivially, for any* $t \in \mathcal{L}^W_{\mathrm{T}}$*,* $\delta_{\underline{a}}(t) = \delta_{\underline{a}}(t)$ *and* (I1) *is valid just in case, for arbitrary model* $\mathcal{M}$*,* $w \in W$ *and* $\underline{a}$*:*

$$\mathcal{M}, w, \underline{\mathrm{a}} \vDash_c t = t \text{ iff } \delta_{\underline{\mathrm{a}}}(t) = \delta_{\underline{\mathrm{a}}}(t)$$

(I2) *is valid iff, for arbitrary model* $\mathcal{M}$*, world* w *and assignment* $\underline{a}$*:*

$$\mathcal{M}, w, \underline{\mathrm{a}} \vDash_c t_1 = t_2 \rightarrow (\phi[t_1/x] \leftrightarrow \phi[t_2/x])$$

If $\mathcal{M}, w, \underline{a} \vDash_c t_1 = t_2$*, then* $\delta_{\underline{a}}(t_1) = \delta_{\underline{a}}(t_2)$*.* $\mathcal{M}, w, \underline{a} \vDash_c \phi[t_1/x] \leftrightarrow \phi[t_2/x]$ *follows by routine induction on the complexity of* ϕ*.*

PLURALITIES. (NE) *is valid on the semantics, since* $\delta_{\underline{a}}(xx)$*, for any* xx *is a non-empty* $d_s \subseteq D_i(w)$*.* (P1) *is valid since* $\mathcal{M}, w, \underline{a} \vDash_c t \prec tt$ *iff* $\delta_{\underline{a}}(t) \in \delta_{\underline{a}}(tt)$ *which holds, if at all, independently of the world parameter.* (Ext) *is valid, since, if* $\mathcal{M}, w, \underline{a} \vDash_c \forall x(x \prec xx \leftrightarrow x \prec yy)$*, for arbitrary* xx *and* yy*, then* $\delta_{\underline{a}}(xx) = \delta_{\underline{a}}(yy)$*. We then show that* $\mathcal{M}, w, \underline{a} \vDash_c \phi(xx) \leftrightarrow \phi(yy)$*, assuming* $\delta_{\underline{a}}(xx) = \delta_{\underline{a}}(yy)$*, by induction on the complexity of* ϕ*.* (Comp) *is valid iff, for arbitrary model* $\mathcal{M}$*, world* w *and assignment* $\underline{a}$*:*

$$\textit{If } \mathcal{M}, w, \underline{a} \vDash_c \exists x \phi(x) \textit{ then } \mathcal{M}, w, \underline{a} \vDash_c \exists xx \forall y(y \prec xx \leftrightarrow \phi(y))$$

$\mathcal{M}, w, \underline{a} \vDash_c \exists xx \forall y(y \prec xx \leftrightarrow \phi(y))$ *iff, for some non-empty* $d_s \subseteq D_i(w)$*:*

(i) *For any* $d \in D_i(w)$*:* $d \in d_s \leftrightarrow d \in v(\phi)_w$*.*

Let $v(\phi)_w$ *be the set* $d_s \subseteq D_i(w)$ *such that each* $y \in d_s$ *iff* $\mathcal{M}, w, \underline{a}[x/y] \vDash_c \phi(x)$*. If* $\mathcal{M}, w, \underline{a} \vDash_c \exists x \phi(x)$*, then* $v(\phi)_w$ *is non-empty. Letting* $d_s = v(\phi)_w$*, we satisfy* (i)*. Finally,* (PE) *is valid, since, if* $\delta_{\underline{a}}(xx) \in D_i(w)$*, for any* $\underline{a}$ *and* w*, then if* $\delta_{\underline{a}}(y) \in \delta_{\underline{a}}(xx)$*, it follows that* $\delta_{\underline{a}}(y) \in D_i(w)$*.*

PROPOSITIONS. (T[]) *is valid, since, by* (35), *for arbitrary* $\mathcal{M}$, w, $\underline{a}$*:*

$$\begin{aligned}\mathcal{M}, w, \underline{\mathrm{a}} \vDash_c \mathrm{T}[\phi] &\textit{ iff } w \rhd \delta_{\underline{\mathrm{a}}}([\phi]) \\ &\textit{ iff } w \in ts_{\underline{\mathrm{a}}}(\phi) \\ &\textit{ iff } \mathcal{M}, w, \underline{\mathrm{a}} \vDash \phi\end{aligned}$$

($\overline{\mathrm{T}}$) *is valid, since* $\mathcal{M}, w, \underline{a} \vDash_c \overline{\mathrm{T}}pp$ *iff each* $\langle \alpha, \beta\rangle \in \delta_{\underline{a}}(pp)$ *is such that* $w \rhd \langle \alpha, \beta\rangle$ *and, if* $\delta_{\underline{a}}(p) \in \delta_{\underline{a}}(pp)$*,* (*i.e. if* $\mathcal{M}, w, \underline{a} \vDash_c p \prec pp$), *then* $w \rhd \delta_{\underline{a}}(p)$ (i.e. then $\mathcal{M}, w, \underline{\mathrm{a}} \vDash_c \mathrm{T}p$). (PNeg1) *is valid, since for arbitrary* $\mathcal{M}$, w, *and* $\underline{a}$*:*

$$\begin{aligned}\mathcal{M}, w, \underline{\mathrm{a}} \vDash_c \mathrm{T}{\sim}p &\textit{ iff } w \rhd \delta_{\underline{\mathrm{a}}}({\sim}p) \\ &\textit{ iff } w \in \alpha : \langle \alpha, \beta\rangle = \delta_{\underline{\mathrm{a}}}({\sim}p) \\ &\textit{ iff } w \notin (W - \alpha) \\ &\textit{ iff } \neg(w \rhd \delta_{\underline{\mathrm{a}}}(p)) \\ &\textit{ iff } \mathcal{M}, w, \underline{\mathrm{a}} \vDash_c \neg\mathrm{T}p\end{aligned}$$

(PNeg2) *is valid, since for arbitrary* $\mathcal{M}$, w, *and* $\underline{a}$:

$$\begin{aligned}\mathcal{M}, w, \underline{\mathrm{a}} \vDash_c p = [\phi] &\textit{ iff } \delta_{\underline{\mathrm{a}}}(p) = \langle ts_{\underline{\mathrm{a}}}(\phi), es_{\underline{\mathrm{a}}}(\phi)\rangle \\ &\textit{ iff } \delta_{\underline{\mathrm{a}}}({\sim}p) = \langle W - ts_{\underline{\mathrm{a}}}(\phi), es_{\underline{\mathrm{a}}}(\phi)\rangle \\ &\textit{ iff } \delta_{\underline{\mathrm{a}}}({\sim}p) = \delta_{\underline{\mathrm{a}}}([\neg\phi]) \\ &\textit{ iff } \mathcal{M}, w, \underline{\mathrm{a}} \vDash_c {\sim}p = [\neg\phi]\end{aligned}$$

□

Next we define a specified class of models $\mathbb{M}^P$ and $\vDash^P_c$—the notion of truth in a $\mathbb{M}^P$–model. The intention is for $\mathbb{M}^P$ to be the class of models of $\mathbb{M}$ in which (P) is valid. So we define $\mathbb{M}^P$ as such, as well as specifying a restriction on the existence sets of the ordered pairs representing propositions.[30] We then define another class of models $\mathbb{M}^@$ and $\vDash^@_c$—the notion of truth in a $\mathbb{M}^@$ model. The semantics defined on $\mathbb{M}^@$ is proven to be sound for the proof system $\vdash^@_c$.

[30] Defining $\mathbb{M}^P$ (in part) as the class of models in which (P) is valid follows the analogous trick of restricting the class of Henkin structures for second-order logic to those which satisfy the Axiom Scheme of Comprehension, see (Walsh and Button, 2018: 25–26) for more details.

Definition 36 ($\mathbb{M}^P$) *Let $\mathbb{M}^P$ be the subclass of $\mathbb{M}$ such that, for $\mathcal{M} \in \mathbb{M}^P$:* (i) $\langle \alpha, \beta \rangle \in D_p(w) \rightarrow w \in \beta$; (ii) $\mathcal{M} \vDash_c \exists p(p = [\phi^{t_1,\ldots,t_n}]) \leftrightarrow \mathrm{E}/\bar{\mathrm{E}}t_1 \wedge \ldots \wedge \mathrm{E}/\bar{\mathrm{E}}t_n$.

Definition 37 ($\vDash_c^P$) *We say that $\vDash_c^P \phi$ iff $\mathcal{M} \vDash_c \phi$, for any $\mathcal{M} \in \mathbb{M}^P$.*

Theorem 38 (Soundness): *If $\vdash_c \phi$, then $\vDash_c^P \phi$, for any formula $\phi \in \mathcal{L}_{\bar{\mathrm{T}}}^W$.*

Proof. Immediate, given (35)–(37). □

Definition 39 ($\vdash_c^@$) *Let $\vdash_c^@$ be the proof system in $\mathcal{L}_{\bar{\mathrm{T}}@}^W$ identical to $\vdash_c$ supplemented with the following rules of inference and axioms.*

(@1) ϕ / $@\phi$ (@2) $@(\phi \rightarrow \psi)$ / $@\phi \rightarrow @\psi$

(@3) $\vdash_c^@ @\neg\phi \leftrightarrow \neg @\phi$ (@4) $\vdash_c^@ @\phi \leftrightarrow \Box @\phi$

Definition 40 ($\mathbb{M}^@$) *Let $\mathbb{M}^@$ be a class of models $\mathcal{M} = \langle W, D_i, D_p, v, w* \rangle$, where W, D_i, D_p, and v are defined by* (31)–(33) *and $w* \in W$.*

Definition 41 ($\vDash_c^@$) *Let $\vDash_c^@$ be an extension of $\vDash_c^P$ with the following principle, where $\mathcal{M} \in \mathbb{M}^@$, for arbitrary $w \in W$, and assignment $\underline{a}$.*

(i) $\mathcal{M}, w, \underline{a} \vDash_c^@ @\phi$ *iff* $\mathcal{M}, w*, \underline{a} \vDash_c^@ \phi$

Theorem 42 (SOUNDNESS $\vdash_c^@$) *If $\vdash_c^@ \phi$, then $\vDash_c^@ \phi$, for any formula $\phi \in \mathcal{L}_{\bar{\mathrm{T}}@}^W$.*

Proof. *Given* (31)–(39)*, it suffices to show that $\vdash_c^@$ is sound for the semantics $\vDash_c^@$ by showing that* (@1)–(@4) *are either logical truths of $\vDash_c^@$ or preserve truth. In the following $\mathcal{M}, w \in W$ and $\underline{a}$ are arbitrary.*

$$\begin{aligned} \vDash_c^@ \phi \ \ &\textit{then}, \mathcal{M}, w, \underline{a} \vDash_c^@ \phi \\ &\textit{then}, \mathcal{M}, w*, \underline{a} \vDash_c^@ \phi \\ &\textit{then}, \vDash_c^@ @\phi \end{aligned} \tag{@1}$$

$$\models^{@}_{c} @(\phi \to \psi) \text{ and } \models^{@}_{c} @\phi \textit{ then } \mathcal{M}, w*, \underline{a} \models^{@}_{c} \phi \to \psi \text{ and } \mathcal{M}, w*, \underline{a} \models^{@}_{c} \phi$$
$$\textit{then } \mathcal{M}, w*, \underline{a} \models^{@}_{c} \psi \tag{@2}$$
$$\textit{then } \mathcal{M}, w, \underline{a} \models^{@}_{c} @\psi$$
$$\textit{then } \mathcal{M}, w, \underline{a} \models^{@}_{c} @\phi \to @\psi$$

$$\models^{@}_{c} @\neg\phi \textit{ iff } \mathcal{M}, w*, \underline{a} \models^{@}_{c} \neg\phi$$
$$\textit{iff } \mathcal{M}, w*, \underline{a} \not\models^{@}_{c} \phi \tag{@3}$$
$$\textit{iff } \mathcal{M}, w, \underline{a} \not\models^{@}_{c} @\phi$$
$$\textit{iff } \models^{@}_{c} \neg @\phi$$

For (@4), *note that if* $\mathcal{M}, w, \underline{a} \models^{@}_{c} @\phi$ *holds, it holds independently of the world parameter. The converse follows from the validity of* (T). □

Next we show that $\mathbb{M}^{P}$ and $\mathbb{M}^{@}$ are non-empty classes of models in our semantics and thus $\vdash_c$ and $\vdash^{@}_{c}$ are consistent.

Theorem 43 $\vdash_c$ *is consistent.*

Proof. Let W *and* X *be two distinct non-empty sets. Let* $\mathcal{M} = \langle W, D_i, D_p, v\rangle$, *where* $D_i(w) = X$ *and* $D_p(w) = \{\langle z, W\rangle : z \subseteq W\}$, *for any* $w \in W$*; and* v *is any valuation function satisfying* (32)(i)–(ii). (33)(i) *is satisfied by construction of* $\mathcal{M}$. (P) *is vacuously true in* $\mathcal{M}$. *Thus* 33(ii) *is satisfied.* □

Theorem 44 $\vdash^{@}_{c}$ *is consistent.*

Proof. Consider the model $\langle W, D_i, D_p, v, w*\rangle$ *which is an extension of* $\mathcal{M} = \langle W, D_i, D_p, v\rangle$ *in the proof of* (43), *where* $w* \in W$. (33)(i) *and* (P) *are satisfied, given the construction.* (@1)–(@4) *are satisfied, given* (42). □

Finally, we prove (24) and (28). Before we establish these, we fix some symbolism and introduce the notion of an automorphism.[31]

[31] The use of automorphisms follows the work in (Fine, 1977b) and (Fine, 1980) and closely follows the work in (Fritz and Goodman, 2016).

Definition 45: *An* automorphism *on* $\mathcal{M} = \langle W, D_i, D_p, v\rangle$ *is a pair* $\langle f, g\rangle$ *of structure preserving permutations. Here,* f *is a permutation of* W*, i.e. a bijection from* W *to itself, and* g *is a permutation on* D_i^A*. Let* $\underline{a}^{\langle f,g\rangle}$ *be the pointwise application of* $\langle f, g\rangle$ *to assignment* $\underline{a}$*. This is defined as follows. Let* $\mathrm{aut}^g(x) = \{g(y) : y \in x\}$ *and* $\mathrm{aut}^f(x) = \{f(y) : y \in x\}$*, where* x *is any set. For each sort of term in* $\mathcal{L}_{\mathrm{T}}^{W}$*, we have:*

(i) (t): $\underline{a}^{\langle f,g\rangle}(t) = g(\underline{a}(t))$

(ii) (tt): $\underline{a}^{\langle f,g\rangle}(tt) = \mathrm{aut}^g(\underline{a}(tt))$

(iii) (p): $\underline{a}^{\langle f,g\rangle}(p) = \langle \mathrm{aut}^f(\alpha), \mathrm{aut}^f(\beta)\rangle$, where $\underline{a}(p) = \langle \alpha, \beta\rangle$

(iv) (pp): $\underline{a}^{\langle f,g\rangle}(pp) = \{\langle \mathrm{aut}^f(\alpha), \mathrm{aut}^f(\beta)\rangle : \langle \alpha, \beta\rangle \in \underline{a}(pp)\}$

An automorphism $\langle f, g\rangle$ *is* structure-preserving *iff the following holds, where* x *is a term of any sort, singular or plural.*

(a) $\underline{a}(x) \in v(F)_w$ iff $\underline{a}^{\langle f,g\rangle}(x) \in v(F)_{f(w)}$

(b) $\underline{a}(x) \in D_i(w)$ iff $\underline{a}^{\langle f,g\rangle}(x) \in D_i(f(w))$

(c) $\underline{a}(x) \in D_p(w)$ iff $\underline{a}^{\langle f,g\rangle}(x) \in D_p(f(w))$

Lemma 46: *If* $\langle f, g\rangle$ *is an automorphism on* $\mathcal{M} = \langle W, D_i, D_p, v\rangle$*, then for any formula* $\phi \in \mathcal{L}_{\mathrm{T}}^{W}$*,* $w \in W$ *and* $\underline{a}$*:*

$$\mathcal{M}, w, \underline{a} \vDash_c \phi \textit{ iff } \mathcal{M}, f(w), \underline{a}^{\langle f,g\rangle} \vDash_c \phi$$

Proof. By induction on the complexity of ϕ*.* □

To prove each of (24) and (28), we just need to define two models, $\mathcal{M}_1 \in \mathbb{M}^{\mathrm{P}}$ and $\mathcal{M}_{@} \in \mathbb{M}^{@}$.

Definition 47 ($\mathcal{M}_1$**)** *For convenience, in the following, we let* $\ulcorner\langle n_1 n_2 ..., m_1 m_2 ...\rangle\urcorner$ *stand for* $\ulcorner\langle \{n_1, n_2, ...\}, \{m_1, m_2, ...\}\rangle\urcorner$*. Let* $\mathcal{M}_1 = \langle W, D_i, D_p, v\rangle$*, where* $W = \{1, 2, 3, 4\}$*,* $D_i(1) = \varnothing$*,*

$D_i(2) = \{5,6\}$, $D_i(3) = \{5,7\}$, *and* $D_i(4) = \{6,7\}$. *Let* v *be the valuation function such that* $v(F)_w = D_i(w)$, *for any non-logical predicate* $F \in \mathcal{L}^W_{\mathrm{T}}$. D_p *is specified:*

$$D_p(1) = \{\langle 1, W\rangle, \langle 234, W\rangle, \langle W, W\rangle, \langle \varnothing, W\rangle\}$$

$$D_p(2) = D_p(1) \cup \{\langle x, 2\rangle | x \subseteq W\} \cup \{\langle x, 23\rangle | x \subseteq W\} \cup \{\langle x, 24\rangle | x \subseteq W\}$$

$$D_p(3) = D_p(1) \cup \{\langle x, 3\rangle | x \subseteq W\} \cup \{\langle x, 23\rangle | x \subseteq W\} \cup \{\langle x, 34\rangle | x \subseteq W\}$$

$$D_p(4) = D_p(1) \cup \{\langle x, 4\rangle | x \subseteq W\} \cup \{\langle x, 24\rangle | x \subseteq W\} \cup \{\langle x, 34\rangle | x \subseteq W\}$$

Definition 48 ($\mathcal{M}_@$) *Let* $\mathcal{M}_@ = \langle W, D_i, D_p, v, w*\rangle$, *where* W, D_i, D_p, v *are as defined in* (49) *and let* $w* = 1$

Theorem 24 *For some* $\mathcal{M} \in \mathbb{M}^P$: $\mathcal{M}, w, \underline{a} \vDash_c \mathrm{ID}$

Proof. *We show* (a) $\mathcal{M}_1 \in \mathbb{M}^P$ *and that* (b) *for some* $w \in W$ *and assignment* $\underline{a}$: $\mathcal{M}_1, w, \underline{a} \vDash_c \Diamond\exists x \Diamond\exists y(\Diamond x \approx_e y \wedge \Diamond(\mathrm{E}x \wedge \neg \mathrm{E}y))$. *First,* (a) *holds iff*

(i) $\langle \alpha, \beta\rangle \in D_p(w) \rightarrow w \in \beta$

(ii) $\mathcal{M}_1 \vDash_c \exists p(p = [\phi^{t_1,\ldots,t_n}]) \leftrightarrow \mathrm{E}/\bar{\mathrm{E}}t_1 \wedge \ldots \wedge \mathrm{E}/\bar{\mathrm{E}}t_n$, *for arbitrary* $\phi^{t_1,\ldots,t_n}$.

By construction of the model, we can see that (i) *holds. For convenience, let* $\ulcorner \delta_{\underline{a}}(t) \gg D(w)\urcorner$ *holds just in case either* $\delta_{\underline{a}}(t) \in D_i(w)$, $\delta_{\underline{a}}(t) \in D_p(w)$, $\delta_{\underline{a}}(t) \subseteq D_i(w)$, *or* $\delta_{\underline{a}}(t) \subseteq D_p(w)$. *Thus* (ii) *holds iff*

(ii*) $\delta_{\underline{a}}([\phi^{t_1,\ldots,t_n}]) \in D_p(w)$ *iff each* t_i *in* $t_1, \ldots, t_n$ *is such that* $\delta_{\underline{a}}(t_i) \gg D(w)$

The left-to-right direction of (ii*) *follows from* (i). *The right-to-left direction of* (ii*) *is established by going through two cases, where* $w = 1, 2$, *establishing* $w = 3, 4$, *by symmetry with* $w = 2$.

CASE I (w = 1): *Suppose, for some* $\underline{a}$ *and* $t_1, \ldots, t_n$ *that each* t_i *in* $t_1, \ldots, t_n$ *is such that* $\delta_{\underline{a}}(t_i) \gg D(1)$. *By construction of* $\mathcal{M}_1$: *if* $\delta_{\underline{a}}(t_i) \gg D(1)$, *then, for any* $w \in W$, $\delta_{\underline{a}}(t_i) \gg D(w)$. *Thus* $\delta_{\underline{a}}([\phi^{t_1,\ldots,t_n}])$ *is some* $\langle \alpha, W\rangle$, *for* $\alpha \subseteq W$. *We define two automorphisms on* $\mathcal{M}_1$, $\langle f, g\rangle$ *and* $\langle f', g'\rangle$ *:*

$\langle f, g\rangle$: $f(1) = 1$; $f(2) = 4$; $f(3) = 3$; and $f(4) = 2$;
$g(5) = 7$; $g(6) = 6$; and $g(7) = 5$.

$\langle f', g'\rangle$ $f'(1) = 1; f'(2) = 2; f'(3) = 4;$ and $f'(4) = 3;$
$g'(5) = 6; g'(6) = 5;$ and $g'(7) = 7.$

Given the specification of $\langle f, g\rangle$ *and* $\langle f', g'\rangle$*, it follows that, for any* t_i *such that* $\delta_{\underline{a}}(t_i) \gg D(1)$*:* $\underline{a}^{\langle f,g\rangle}(t_i) = \underline{a}(t_i)$ *and* $\underline{a}^{\langle f',g'\rangle}(t_i) = \underline{a}(t_i)$*. Thus, given* (46)*, and the fact that* $f(2) = 4$ *and* $f'(3) = 4$ *it follows that, for any* $\phi^{t_1,...,t_n} \in \mathcal{L}_{\mathrm{T}}^{W}$ *where all* t_i *in* $t_1, .., t_n$ *are such that* $\delta_{\underline{a}}(t_i) \gg D(1)$*:*

$$(1)\ \mathcal{M}_1, 2, \underline{a} \vDash_c^{@} \phi^{t_1,...,t_n} \text{ iff } \mathcal{M}_1, 4, \underline{a} \vDash_c^{@} \phi^{t_1,...,t_n}$$
$$(2)\ \mathcal{M}_1, 3, \underline{a} \vDash_c^{@} \phi^{t_1,...,t_n} \text{ iff } \mathcal{M}_1, 4, \underline{a} \vDash_c^{@} \phi^{t_1,...,t_n}$$

From (1) *and* (2)*, it follows that all* $\langle \alpha, \beta\rangle \in D_p(1)$ *are such that* $2 \in \alpha$ *iff* $3 \in \alpha$ *iff* $4 \in \alpha$*. Thus, if all* t_i *in* $t_1, .., t_n$ *are such that* $\delta_{\underline{a}}(t_i) \gg D(1)$*, then:*

$$(3)\ \delta_{\underline{a}}([\phi^{t_1,...,t_n}]) \in D_p(1) \text{ iff } \Big(2 \rhd \delta_{\underline{a}}([\phi^{t_1,...,t_n}]) \text{ iff } 3 \rhd \delta_{\underline{a}}([\phi^{t_1,...,t_n}]) \text{ iff } 4 \rhd \delta_{\underline{a}}([\phi^{t_1,...,t_n}])\Big)$$

The left-to-right direction of (3) *is immediate. The right-to-left direction of* (3) *follows from the fact that if all* t_i *in* $t_1, .., t_n$ *are such that* $\delta_{\underline{a}}(t_i) \gg D(1)$ *and* $2 \rhd \delta_{\underline{a}}([\phi^{t_1,...,t_n}])$ *iff* $3 \rhd \delta_{\underline{a}}([\phi^{t_1,...,t_n}])$ *iff* $4 \rhd \delta_{\underline{a}}([\phi^{t_1,...,t_n}])$*, then* $\delta_{\underline{a}}([\phi^{t_1,...,t_n}]) = \langle \alpha, W\rangle$*, where either* $2, 3, 4 \in \alpha$ *or* $2, 3, 4 \notin \alpha$*. The set of* $\langle \alpha, W\rangle$ *such that either* $2, 3, 4 \in \alpha$ *or* $2, 3, 4 \notin \alpha$ *is*

$$D* = \{\langle 1, W\rangle, \langle 234, W\rangle, \langle W, W\rangle, \langle \varnothing, W\rangle\}$$

By inspection, $D_p(1) = D*$ *and thus* $\delta_{\underline{a}}([\phi^{t_1,...,t_n}]) \in D_p(1)$*, for arbitrary* $\phi^{t_1,...,t_n}$ *and* $\underline{a}$*.*

CASE II (w = 2): *We establish* (ii) *when* $w = 2$*, for arbitrary* $\phi^{t_1,...,t_n} \in \mathcal{L}_{\mathrm{T}}^{W}$ *and* $\underline{a}$ *using the following facts.*

1. *By* (i) *each* t_i *in* $t_1, ..., t_n$ *is such that* $\delta_{\underline{a}}(t_i) \gg D(2)$*.*
2. *By the construction of* $\mathcal{M}_1$*, it follows that*
   a. *If* $\delta_{\underline{a}}([\phi^{t_1,...,t_n}]) \in D_p(1)$*, then, all* $w \in W$*:* $\delta_{\underline{a}}([\phi^{t_1,...,t_n}]) \in D_p(w)$*.*
   b. *If* $\delta_{\underline{a}}([\phi^{t_1,...,t_n}])$ *is some* $\langle \alpha, \beta\rangle$ *such that* $2, 3, 4 \in \beta$*, then* $1 \in \beta$*.*

It follows from (1.) *and* (2.) *that if* $\delta_{\underline{a}}([\phi^{t_1,...,t_n}]) \in D_p(2)$*, then* $\delta_{\underline{a}}([\phi^{t_1,...,t_n}]) = \langle \alpha, \beta\rangle$*, where* $\alpha \subseteq W$ *and either* $\beta = W$*,* $\beta = \{2\}$*,* $\beta = \{2, 3\}$*,* $\beta = \{2, 4\}$*. We can see that, by construction of* $\mathcal{M}_1$*, for any* $\alpha \subseteq W$*:* $\langle \alpha, 2\rangle \in D_p(2)$*,* $\langle \alpha, 23\rangle \in D_p(2)$*, and* $\langle \alpha, 24\rangle \in D_p(2)$*. Moreover, given* CASE I*, for any* $\langle \alpha, W\rangle$ *such that* $2 \in \alpha$ *iff* $3 \in \alpha$*,* $\langle \alpha, W\rangle \in D_p(2)$*. Thus* $\delta_{\underline{a}}([\phi^{t_1,...,t_n}]) \in D_p(2)$*, for arbitrary* $\phi^{t_1,...,t_n}$*.*

Cases of $w = 3, 4$ are established by symmetry with CASE II.

Next, (b)*, i.e. for some $w \in W$ and $\underline{a}$: $\mathcal{M}_1, w, \underline{a} \vDash_c \Diamond\exists x\Diamond\exists y(\Diamond x \approx_e y \wedge \Diamond(\mathrm{E}x \wedge \neg\mathrm{E}y))$. For this we show that, for some $d_s \in D_i(2)$, some $w' \in W$, and $d_s* \in D_i(w')$:*

(i) $\mathcal{M}_1, w', \underline{\mathrm{a}}[x/d_s, y/d_s*] \vDash_c \Diamond x \approx_e y$

(ii) $\mathcal{M}_1, w', \underline{\mathrm{a}}[x/d_s, y/d_s*] \vDash_c \Diamond(\mathrm{E}x \wedge \neg\mathrm{E}y)$.

Let $w' = 2$, $d_s = 5$, and $d_s* = 6$. (i) *holds iff for some $w'' \in W$:*

$$\mathcal{M}_1, w'', \underline{a}[x/d_s, y/d_s*] \vDash_c \forall pp(\Box(\bar{\mathrm{T}}pp \to \mathrm{E}x) \leftrightarrow \Box(\bar{\mathrm{T}}\mathrm{pp} \to \mathrm{E}y))$$

Let $w'' = 1$, $d_s = 5$, and $d_s = 6$. Thus,* (i) *holds if for any $d_s' \subseteq D_p(1)$:*

$$\text{(iii) } \mathcal{M}_1, 1, \underline{\mathrm{a}}[x/5, y/6, pp/d_s'] \vDash_c \Box(\bar{\mathrm{T}}pp \to \mathrm{E}x) \leftrightarrow \Box(\bar{\mathrm{T}}\mathrm{pp} \to \mathrm{E}y).$$

There are fifteen non-empty $d_s' \subseteq D_p(1)$. We immediately note that eight such $d_s' \subseteq D_p(1)$ are such that $\langle\varnothing, W\rangle \in d_s'$ and so trivially satisfy (iii)*. Two further subsets of $D_p(1)$, namely $\{\langle 1, W\rangle, \langle 234, W\rangle\}$ and $\{\langle 1, W\rangle, \langle 234, W\rangle, \langle W, W\rangle\}$, also trivially satisfy* (iii*)*. Grouping common cases, we have the following two:

CASE I: (*Either $d_s' = \{\langle 1, W\rangle\}$, $d_s' = \{\langle W, W\rangle\}$ or $d_s' = \{\langle 1, W\rangle, \langle W, W\rangle\}$.) In each case, $\mathcal{M}_1, 1, \underline{a}[x/5, y/6, pp/d_s'] \vDash_c (\bar{\mathrm{T}}pp \wedge \neg\mathrm{E}x) \wedge (\bar{\mathrm{T}}pp \wedge \neg\mathrm{E}y)$. Thus, it follows: $\mathcal{M}_1, 1, \underline{a}[x/5, y/6, pp/d_s'] \vDash_c \Diamond(\bar{\mathrm{T}}pp \wedge \neg\mathrm{E}x) \wedge \Diamond(\bar{\mathrm{T}}pp \wedge \neg\mathrm{E}y)$ and thus* (iii)*.*

CASE II: (*Either $d_s' = \{\langle 234, W\rangle\}$ or $d_s' = \{\langle 234, W\rangle, \langle W, W\rangle\}$.) In both cases, it follows that* (i) *$\mathcal{M}_1, 4, \underline{a}[x/5, y/6, pp/d_s'] \vDash_c \bar{\mathrm{T}}pp \wedge \neg\mathrm{E}x$ and it follows that* (ii) *$\mathcal{M}_1, 3, \underline{a}[x/5, y/6, pp/d_s'] \vDash_c \bar{\mathrm{T}}pp \wedge \neg\mathrm{E}y$. From* (i) *and* (ii) *it follows that $\mathcal{M}_1, 1, \underline{a}[x/5, x/6, pp/d_s'] \vDash_c \Diamond(\bar{\mathrm{T}}pp \wedge \neg\mathrm{E}x) \wedge \Diamond(\bar{\mathrm{T}}pp \wedge \neg\mathrm{E}y)$. Thus* (iii) *holds.*

Thus (iii) and thus (i). (ii) holds given the following: $\mathcal{M}_1, 3, \underline{a}[x/5, x/6, pp/d_s'] \vDash_c \mathrm{E}x \wedge \neg\mathrm{E}y$. Therefore: $\mathcal{M}_1, 2, \underline{\mathrm{a}}[x/5, x/6, pp/d_s'] \vDash_c \Diamond(\mathrm{E}x \wedge \neg\mathrm{E}y)$. □

Theorem 28 *For some* $\mathcal{M} \in \mathbb{M}^{@}$*:* $\mathcal{M}, w, \underline{a} \vDash_c^{@}$ ID@

Proof. We use $\mathcal{M}_{@}$*.* $\mathcal{M}_{@}, 1, \underline{a} \vDash_c^{@} \Diamond\exists x \exists y(@x \approx_e y \wedge \Diamond(\mathrm{E}x \wedge \neg \mathrm{E}y))$ *holds iff, for some* $w' \in W$*, assignment* $\underline{a}$ *and some* $d_s, d_s* \in D_i(w')$*:*

(1) $\mathcal{M}_{@}, w', \underline{\mathrm{a}}[x/d_s, y/d_s*] \vDash_c^{@} @x \approx_e y$

(2) $\mathcal{M}_{@}, w', \underline{\mathrm{a}}[x/d_s, y/d_s*] \vDash_c^{@} \Diamond(\mathrm{E}x \wedge \neg \mathrm{E}y)$

Let $w' = 2$*,* $d_s = 5$*, and* $d_s* = 6$*. Thus* (1) *and* (2) *hold if, respectively:*

(1′) $\mathcal{M}_{@}, 2, \underline{\mathrm{a}}[x/5, x/6] \vDash_c^{@} @x \approx_e y$

(2′) $\mathcal{M}_{@}, 2, \underline{\mathrm{a}}[x/5, x/6] \vDash_c^{@} \Diamond(\mathrm{E}x \wedge \neg \mathrm{E}y)$

Given that $w* = 1$ *it follows from* $\mathcal{M}_{@}, 1, \underline{a}[x/5, x/6] \vDash_c^{@} x \approx_e y$ *that* (1′) *and thus* (1). $\mathcal{M}_{@}, 3, \underline{a}[x/5, x/6] \vDash_c^{@}$ $\mathrm{E}x \wedge \neg \mathrm{E}y$ *holds. Thus* (2′) *and thus* (2). □

Finally I address the point in fn. 26. Namely, that the *unactualised* fundamental theorem for possibility, i.e. $\phi \leftrightarrow \exists pp(\mathrm{W}^{\mathcal{C}}pp \wedge pp \models [\phi])$, for any $\phi \in \mathcal{L}_{\mathrm{T}}^{W}{}_{@}$ with no free *pp*, assuming that $\mathrm{W}^{\mathcal{C}}$ satisfies (W), is not *real world valid*.

Definition 49 (Real World Validity) *Formula* $\phi \in \mathcal{L}_{\mathrm{T}}^{W}{}_{@}$ *is* real world valid *just in case for any* $\mathcal{M} \in \mathbb{M}^{@}$ *and assignment* $\underline{a}$*:* $\mathcal{M}, w*, \underline{a} \vDash_c^{@} \phi$*.*

Theorem 50 *It is not the case that, for any* $\phi \in \mathcal{L}_{\mathrm{T}}^{W}{}_{@}$ *with no free pp:* $\phi \leftrightarrow \exists pp(\mathrm{W}^{\mathcal{C}}pp \wedge pp \models [\phi])$ *is real world valid assuming that* $\mathrm{W}^{\mathcal{C}}$ *satisfies* (W)

Proof. Given (27) *and the proof of* (28) *above:* $\mathcal{M}_{@}, 1, \underline{a} \vDash_c^{@} \neg @\exists pp(\mathrm{W}(pp \wedge pp \models [\mathrm{E}x \wedge \neg \mathrm{E}y]))$*. Thus:* $\mathcal{M}_{@}, 1, \underline{a} \vDash_c^{@} @\neg\exists pp(\mathrm{W}(pp \wedge pp \models [\mathrm{E}x \wedge \neg \mathrm{E}y]))$ *and* $\mathcal{M}_{@}, 1, \underline{a} \vDash_c^{@} \Diamond(\mathrm{E}x \wedge \neg \mathrm{E}y)$*. Since* $w* = 1$ *in* $\mathcal{M}_{@}$*, it follows that* $\mathcal{M}_{@}, w*, \underline{a} \nvDash_c^{@} \exists pp(\mathrm{W}(pp \wedge pp \models [\mathrm{E}x \wedge \neg \mathrm{E}y]))$ *and* $\mathcal{M}_{@}, w*, \underline{a} \vDash_c^{@} \Diamond(\mathrm{E}x \wedge \neg \mathrm{E}y)$*. Therefore:* $\mathcal{M}_{@}, w*, \underline{a} \nvDash_c^{@} \Diamond(\mathrm{E}x \wedge \neg \mathrm{E}y) \leftrightarrow \exists pp(\mathrm{W}(pp \wedge pp \models [\mathrm{E}x \wedge \neg \mathrm{E}y]))$*.* □

References

Adams, Robert Merrihew (1981). Actualism and Thisness. *Synthese* 49, 3–41.

Bealer, George (1982). *Quality and Concept*. Oxford, England: Oxford University Press.

Bealer, George (1993). Universals. *Journal of Philosophy* 90, 5–32.

Bealer, George (1994). Property Theory: The Type-Free Approach V. The Church Approach. *Journal of Philosophical Logic* 23, 139–171.

Bealer, George (1998). Propositions. *Mind* 107, 1–32.

Bealer, George and Uwe Mönnich (1989). Property Theories. In: *Handbook of Philosophical Logic, Volume IV*. Ed. by Dov Gabbay and Franz Guenthner. Kluwer Academic Publishers, 133–251.

Bringsjord, Selmer (1985). Are there set theoretic possible worlds? *Analysis* 45, 64.

Davies, Martin and Lloyd Humberstone (1980). Two notions of necessity. *Philosophical Studies* 38, 1–30.

Deutsch, Harry (1990). Contingency and Modal Logic. *Philosophical Studies* 60, 89–102.

Einheuser, Iris (2012). Inner and Outer Truth. *Philosophers' Imprint* 12.

Fine, Kit (1977a). Prior on the Construction of Possible Worlds and Instants. In: Fine, Kit and Arthur Prior. *Worlds, Times and Selves*. Duckworth, 116–161.

Fine, Kit (1977b). Properties, Propositions and Sets. *Journal of Philosophical Logic* 6, 135–191.

Fine, Kit (1980). First-order Modal Theories. *Studia Logica* 39, 159–202.

Fine, Kit (1985). Plantinga on the Reduction of Possibilist Discourse. In: *Profiles: Alvin Plantinga*. Ed. by H. Tomberlin and P. van Inwagen. Springer, 145–186.

Fitch, Gregory (1996). In Defense of Aristotelian Actualism. *Philosophical Perspectives* 10, 53–71.

Forrest, Peter (1986). Ways Worlds Could Be. *Australasian Journal of Philosophy* 64, 15–24.

Fritz, Peter (2016). Propositional Contingentism. *Review of Symbolic Logic* 9, 123–142.

Fritz, Peter (2017). Logics for Propositional Contingentism. *Review of Symbolic Logic* 10, 203–236.

Fritz, Peter (2018a). Higher-Order Contingentism, Part 2: Patterns of Indistinguishability. *Journal of Philosophical Logic* 47, 407–418.

Fritz, Peter (2018b). Higher-Order Contingentism, Part 3: Expressive Limitations. *Journal of Philosophical Logic* 47, 649–671.

Fritz, Peter (Unpublished). Being Somehow Without (Possibly) Being Something.

Fritz, Peter and Jeremy Goodman (2016). Higher-order Contingentism, Part 1: Closure and Generation. *Journal of Philosophical Logic* 45, 645–695.

Fritz, Peter and Jeremy Goodman (2017a). Counterfactuals and Propositional Contingentism. *Review of Symbolic Logic* 10, 509–529.

Fritz, Peter and Jeremy Goodman (2017b). Counting Incompossibles. *Mind* 126, 1063–1108.

Fritz, Peter and Nicholas K. Jones (forthcoming). *Higher-Order Metaphysics*. Oxford University Press.

Grim, Patrick (1986). On Sets and Worlds: A Reply to Menzel. *Analysis* 46, 186–191.

Hewitt, Simon (2012). Modalising Plurals. *Journal of Philosophical Logic* 41, 853–875.

Hewitt, Simon (2015). When Do Some Things Form a Set? *Philosophia Mathematica* 23, 311–337.

Ingram, David (2018a). *Thisness Presentism: An Essay on Time, Truth, and Ontology*. Routledge.

Ingram, David (2018b). Thisnesses, Propositions, and Truth. *Pacific Philosophical Quarterly* 99, 442–463.

Jacinto, Bruno (2019). Serious Actualism and Higher-Order Predication. *Journal of Philosophical Logic* 48, 471–499.

Linnebo, Øystein (2006). Sets, Properties, and Unrestricted Quantification. In: *Absolute Generality*. Ed. by Gabriel Uzquiano and Agustin Rayo. Oxford University Press.

Linnebo, Øystein (2016). Plurals and Modals. *Canadian Journal of Philosophy* 46, 654–676.

Linnebo, Øystein (2017). Plural Quantification. In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Metaphysics Research Lab, Stanford University.

McMichael, Alan (1983). A Problem for Actualism about Possible Worlds. *The Philosophical Review* 92, 49–66.

Menzel, Christopher (1986). On Set Theoretic Possible Worlds. *Analysis* 46, 68–72.

Menzel, Christopher (1991). The True Modal Logic. *Journal of Philosophical Logic* 20, 331–374.

Menzel, Christopher (1993). The Proper Treatment of Predication in Fine-Grained Intensional Logic. *Philosophical Perspectives* 7, 61–87.

Menzel, Christopher (2012). Sets and Worlds Again. *Analysis* 72, 304–309.

Menzel, Christopher (forthcoming). Pure Logic and Higher-order Metaphysics. In: *Higher-order Metaphysics*. Ed. by P. Fritz and N. Jones. OUP, Oxford.

Menzel, Christopher and Edward N Zalta (2014). The Fundamental Theorem of World Theory. *Journal of Philosophical Logic* 43, 333–363.

Merricks, Trenton (2015). *Propositions*. Oxford, England: Oxford University Press.

Mitchell-Yellin, Benjamin and Michael Nelson (2016). S5 for Aristotelian Actualists. *Philosophical Studies* 173, 1537–1569.

Myhill, John (1958). Problems arising in the formalization of intensional logic. *Logique et Analyse* 1, 74–83.

Pickel, Bryan (forthcoming). Against Second-Order Primitivism. In: *Higher-order Metaphysics*. Ed. by P. Fritz and N. Jones. OUP, Oxford.

Plantinga, Alvin (1976). Actualism and Possible Worlds. *Theoria* 42, 139–160.

Plantinga, Alvin (1979). *The Nature of Necessity*. Oxford University Press, USA.

Pollock, John L (1985). Plantinga On Possible Worlds. In: *Profiles: Alvin Plantinga*. Springer, 121–144.

Prior, Arthur N (1957). *Time and Modality*. Oxford University Press, Oxford.

Prior, Arthur N (1967). *Past, Present and Future*. Oxford University Press, Oxford.

Russell, Bertrand (1903). *The principles of mathematics*.

Salmon, Nathan (1987). Existence. *Philosophical Perspectives* 1, 49–108.

Skiba, Lukas (2021). Higher-Order Metaphysics. *Philosophy Compass* 16, 1–11.

Speaks, Jeff (2012). On Possibly Nonexistent Propositions. *Philosophy and Phenomenological Research* 85, 528–562.

Stalnaker, Robert (1976). Propositions. In: *Issues in the Philosophy of Language: Proceedings of the 1972 Colloquium in Philosophy*. Ed. by Alfred F. MacKay and Daniel D. Merrill. New Haven and London: Yale University Press, 79–91.

Stalnaker, Robert (2012). *Mere Possibilities: Metaphysical Foundations of Modal Semantics*. Princeton University Press.

Stephanou, Yannis (2007). Serious Actualism. *Philosophical Review* 116, 219–250.

Turner, Jason (2005). Strong and Weak Possibility. *Philosophical Studies* 125, 191–217.

Uzquiano, Gabriel (2011). Plural Quantification and Modality. In: *Proceedings of the Aristotelian Society*. Vol. 3, 219–250.

Walsh, Sean and Tim Button (2018). *Philosophy and Model Theory*. Oxford, UK: Oxford University Press.

Williamson, Timothy (2013). *Modal Logic as Metaphysics*. Oxford University Press, Oxford.