The Nature of Logic, Part II: Philosophical Logic.

Lecture II, *Putnam on Quantum Logic*, 23rd February.

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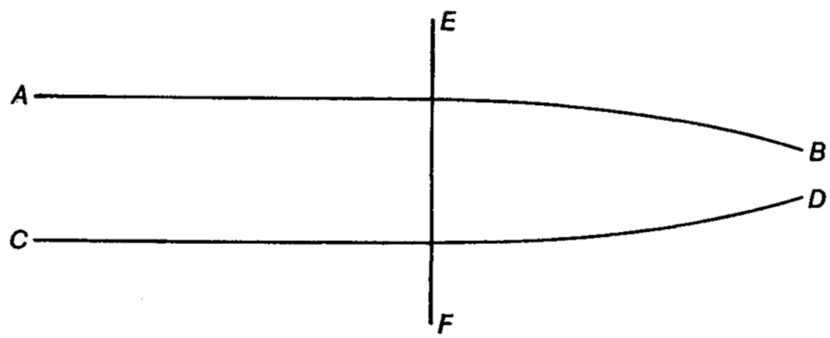
Last week, we started covering some preliminary ground—what is logic and what questions can we ask about its nature— and we then looked at the view that logic is analytic and *a priori*. We looked at one particularly extreme way of rejecting the analytic-*a priori* view of logic—Quine's holistic argument that everything, including logic, is revisable. We looked at some objections to Quine's rejection of the *a priority* of all logic. Our tentative conclusion: sure, we can't revise *all* logic, but that's consistent some logic being *a posteriori*.

This week, we'll look at Putnam's influential proposal for adopting *quantum* logic in (Putnam, 1979) as an interesting case study of the kind of empirical argument one could give for revising logic. This week, will be largely expository. Next week, we'll discuss in more detail whether Putnam's argument is successful.

1. Putnam on Geometry

1.1 To understand Putnam's case for revising classical logic to quantum logic, we should first focus on what he has to say about geometry—particularly, what he has to say about the shift away from Euclidean geometry in the advent of General Relativity. For Putnam, there is a perfect parallel between the case of revising geometry and the case of revising classical logic to quantum logic. (Quantum logic is first introduced in (Birkhoff and Neumann, 1936)).

1.2. Consider two straight lines *AB* and *CD* which are a constant distance apart to the 'left' of a perpendicular line *EF*, but which, after a crossing *EF*, begin to converge. This is illustrated in (Putnam, 1979):



Initially, one might think that the situation here is not possible. At the very least, the situation as we described it might be thought impossible—we cannot have two *straight lines* like this. Whilst this situation is not a formal contradiction of the form $p \wedge \neg p$, intuitively this situation is just as contradictory as saying that a ball is both red all over and blue all over, or someone is bachelor and married, see (Putnam, 1979: 174–6)

1.3. Or is it? What's interesting about this case is that, according to General Relativity, this is not only possible, but *it actually occurs*. That is, one can have two straight lines which are locally parallel, but which converge. In Euclidean geometry, if two lines are straight and locally parallel, then they cannot converge at any point in space. However, according to General Relativity, space-time has a non-Euclidean geometry. In both Euclidean and Non-Euclidean geometry, we have the same definition of a straight line: a straight line between two points is the shortest path between those two points. However, in non-Euclidean geometry, this no longer entails that two locally parallel lines are non-convergent.

1.4. Putnam finds this kind of case striking: it is the overturning of previously 'necessary' truths by a fundamentally empirical argument (1979: 190–98) What's important to our present concerns here is that Putnam argues that we should think that precisely the same thing should happen in *logic*. Given the advent of Quantum Mechanics, we should reject classical in favour of *quantum logic*. Logic is *as empirical as geometry*.

2. A Précis of Quantum Logic

2.1. We should first say what quantum logic *is* before talking about why we should revise classic logic to quantum logic. There are various ways of defining quantum logic. Putnam initially defines quantum logic by giving an *interpretation* of the mathematical formalism of quantum mechanics. This is done with a certain kind of vector space of infinite dimensions—a **Hilbert Space**. According to this interpretation, we interpret sub-spaces of a Hilbert space H as propositions, we introduce various operations on subspaces, e.g., *span* and *intersection*, and these operations correspond to the *logical* operations like conjunction and disjunction.

2.2. An equivalent (and slightly easier) way of defining quantum logic is using the notion of a **lattice**. A lattice is set-theoretic structure. In particular, a special kind of partially ordered set, i.e., members of the set satisfy a relation (a partial ordering) which is reflexive, anti-symmetric, and transitive. A lattice is closed under two operations called *join* ($\vee$) and *meet* ($\wedge$). (See the Appendix for all the details.)

2.3. For our purposes, we can be concrete and just consider the set of propositions P partially ordered by entailment. In this case, we can think of join and meet as corresponding to the disjunction and conjunction of two propositions. So, the set P of propositions partially ordered by entailment indeed forms a lattice. It closed under join and meet: if $p, q \in P$, then $p \vee q \in P$ and $p \wedge q \in P$. Lattices can have further properties:

Ortholattice: A lattice is an *ortholattice* if there is a so-called *greatest* element (1) and a so-called *least* element (0). In the case of the set P, think of the greatest element as the proposition entailed by any $p \in P$. In the case of P, think of the least element as the proposition entailed by no $p \in P$. (Here propositions are coarse-grained and so there is a *unique* least and greatest element.)

Orthocomplementation: A lattice is *orthocomplemented* if there is an operation called complementation $*$ such that, for any two members a, b of the set: (i) $a \vee a^* = 1$; (ii) $a \wedge a^* = 0$; (iii) $a^{**} = a$; and (iv) $(a \wedge b)^* = a^* \vee b^*$. In the case of the set P, think of $*$ as negation.

Distributive: A lattice is distributive if and only if, for any a, b, c in the set: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

2.4. We can now precisely characterise quantum logic and the difference between quantum logic and classical logic. Propositions form a distributive orthocomplemented lattice with respect to entailment, according to classical logic. This is equivalent to saying that propositions form a Boolean algebra, according to classical logic. However, propositions from a *non-distributive* orthocomplemted lattice, according to quantum logic.

2.5. There are significant similarities between distributive and non-distributive orthocomplemented lattices. As such, there are significant similarities between quantum and classic logic. The following hold for both:

($\vee$) $p \vDash p \vee q$; and if $p_1 \vDash q$ and $p_2 \vDash q$, then $p_1 \vee p_2 \vDash q$.

($\wedge$) $p \wedge q \vDash p$; and $p, q \vDash p \wedge q$.

($\neg$) $p \wedge \neg p$ is a contradiction; $\vDash p \vee \neg p$ and $\neg\neg p \vDash p$.

The crucial difference between classical and quantum logic boils down to a disagreement over the standard distribution law, i.e., $p \wedge (q \vee r) \vDash (p \wedge q) \vee (p \wedge r)$. This, on the face of it, seems wholly unproblematic: I'm going to go town and it will either rain or it won't rain *therefore* either I'm going to go to town and it will rain or I'm going to go to town and it will not rain. So, why reject distribution? Putnam's fundamental point is that distribution is at the heart of several deep puzzles arising from quantum mechanics. So, blame distribution.

3. Putnam's Case for Quantum Logic

3.1. Broadly speaking, Putnam's argument for quantum logic is not that it is a more convenient logic with which we should formulate quantum mechanics. Rather, the thought is that if we instead stick with classical logic, we are forced to make conclusions about various quantum phenomena which we should very much avoid. Classical logic should be *banned*, just as Euclidean geometry was ditched after General Relativity.

3.2. Consider the phenomena of complementarity. In quantum mechanics, the uncertainty principle tells us that we cannot specify the precise momentum and precise position of any given particle. That is, if M_1 specifies the precise momentum of, say, an electron and P_1 specifies the precise position of the same electron, then there cannot be a specification $M_1 \wedge P_1$ of the precise momentum and position of the electron. This is puzzling. In fact, this is especially puzzling because quantum mechanics does not rule out a specification of some precise momentum M *alone* for a particle. Moreover, the following is a valid inference, where $P_1 \vee ... \vee P_n$ is the disjunction of all the possible specifications of the precise position of the same particle.

$$M \vDash_{Q/C} M \wedge (P_1 \vee ... \vee P_n) \tag{1}$$

According to classical logic, which contains distribution, we also have:

$$M \wedge (P_1 \vee ... \vee P_n) \vDash_C (M \wedge P_1) \vee ... \vee (M \wedge P_n) \tag{2}$$

So, by (1) and the transitivity of entailment, we get the following valid inference in classical logic:

$$M \vDash_c (M \wedge P_1) \vee ... \vee (M \wedge P_n) \tag{3}$$

But *each* of $(M \wedge P_i)$ is problematic in quantum mechanics. Quantum logic avoids (3), since distribution fails to be quantum logically valid: $M \wedge (P_1 \vee ... \vee P_n) \nvDash_Q (M \wedge P_1) \vee ... \vee (M \wedge P_n)$. Indeed, according to quantum logic, any $(\phi_m \wedge \psi_p)$, where ϕ_m involves specifying the precise momentum of some particle and ψ_m involves specifying the precise position of the same particle is a *logical contradiction* (Putnam, 1979: 180).

3.3. An essential part of Putnam's argument is that the only way of preserving classical logic in the face of it licensing problematic inferences like (3) is to adopt bad positions in the metaphysics of science. For instance, you might try to explain away the issue with (3) by appealing an interpretation of quantum mechanics in which observation or measurement *collapses* the uncertainty over the precise specification of momentum and position, or an interpretation where one of the disjuncts $M \wedge P_i$ is really true, just undetectable. Putnam thinks that none of these options are palatable; or at least more palatable than revising the law of distribution.

3.4. Putnam gives another argument for revising our logic to quantum logic, stemming from a puzzle which arises from the so-called double-slit experiment. Here's the set up for the experiment.

Double-Slit Experiment: We have a vaccuum chamber, some controlled source of photons at one end and a photographic plate at the other end. In between the photon source and the photographic plate is barrier with two equal slits, allowing the photons to pass through to the photographic plate.

3.5. Now, if we let A_1 be 'the photon passes through the first slit', A_2 be 'the photon passes through the second slit', and R be 'the photon strikes a particular tiny region', then we can calculate the probability that the photon strikes R given that it goes through the first slit $Pr(R|A_1)$ and the probability that the photon strikes R given that it goes through the second slit $Pr(R|A_2)$. Importantly, if only the first slit is open, then

$Pr(R|A_1)$ is the same as $Pr(R|A_2)$, if only the second slit is open. Moreover, this fact is both experimentally verifiable and calculable from both quantum and classical mechanics.

3.6. Why is this an important fact? It's important because things get odd when we try to calculate, using classical mechanics, the probability the photon hits region R if we leave both slits open. Classical mechanics predicts that the probability that the photon hits region R is equal to:

$$\frac{1}{2}Pr(R|A_1) + \frac{1}{2}Pr(R|A_2) \tag{4}$$

However, quantum mechanics predicts otherwise. Indeed, (4) is experimentally invalidated. This alone should be startling because the classical derivation of the relevant probabilities when only one slit was open conformed to experiment. What difference should there being two slits open have for each single photon?

3.7. Putnam's diagnosis of the problem here is that there is only a clash between the classical mechanical derivation and the two-slit case if the derivation is carried out in a classical logic. The full derivation of (4) is unnecessary to show this. What's important is that to derive (4), we begin by calculating the probability that the photon hits region R, given that it goes through either the first or the second slit. After all, to hit the region at all, it must go through one or the other. Now, by the standard definition of conditional probability:

$$Pr(R|A_1 \vee A_2) = \frac{Pr((A_1 \vee A_2) \wedge R)}{Pr(A_1 \vee A_2)} \tag{5}$$

And a crucial part of the classical derivation of (4) is the following.

$$\frac{Pr((A_1 \vee A_2) \wedge R)}{Pr(A_1 \vee A_2)} = \frac{Pr((A_1 \wedge R) \vee (A_2 \wedge R))}{Pr(A_1 \vee A_2)} \tag{6}$$

Of course, (6) is true according to *classical logic*. However, (6) holds only because of distribution. So, (6) fails to be true, according to quantum logic. For Putnam, this is crucial. If our logic is classical, then we get a clash between the classical behaviour of the photon in the one-slit case and the non-classical behaviour of the photon in the two-slit case. If our logic is quantum, (4) is not even a *classical* prediction.

3.8. Of course, Putnam is not here saying that classical mechanics is right. Rather, his point is that there is no need for further explanatory work—no need to reach again for bad philosophy of science—to account for the mysterious divergence from the predicted (4) in the two-slit case, see (Putnam, 1979: 181):

> Someone who believes classical logic must conclude from the failure of the classical law that one photon can somehow go through two slits ... or believe that the electron somehow 'prefers' one slit to the other (but only when no detector is placed in the slit to detect this mysterious preference), or believe that in some strange way the electron going through slit 1 'knows' that slit 2 is open and behaves differently than it would if slit 2 were closed; while someone who believe quantum logic would see no reason to predict [(4)] in the first place.

References

Birkhoff, Garrett and John Von Neumann (1936). The Logic of Quantum Mechanics. *Annals of Mathematics* 37, 823–843. ISSN: 0003486X. URL: http://www.jstor.org/stable/1968621 (visited on 02/22/2024).

Putnam, Hilary (1979). The Logic of Quantum Mechanics. In: *Philosophical Papers: Volume 1, Mathematics, Matter and Method*. Ed. by Hilary Putnam. Cambridge University Press.

Appendix: Partial Orders, Lattices, Orthocomplementation, and Distribution.

DEFINITION 1. (Partial Order). *A partial order is a binary relation $\leq$ on a set A if and only if:*

(i) *$\leq$ is reflexive (for any $a \in A$: $a \leq a$)*

(ii) *$\leq$ is anti-symmetric (for any $a, b \in A$: if $a \leq b$ and $b \leq a$, then $a = b$)*

(iii) *$\leq$ is transitive (for any $a, b, c \in A$: if $a \leq b$, $b \leq c$, then $a \leq c$)*

DEFINITION 2. (Poset). *A partially ordered set (poset) is a pair $\langle A, \leq \rangle$, where $\leq$ is a partial ordering on A.*

DEFINITION 3. (Bounds). *If $\langle A, \leq \rangle$ is a partially ordered set and $B \subseteq A$, then:*

(i) *$a \in A$ is an upper bound of B if $b \leq a$, for any $b \in B$*

(ii) *$a \in A$ is a lower bound of B if $a \leq b$, for any $b \in B$*

(iii) *$a \in A$ is the least upper bound of B if a is upper bound of B and $a \leq y$, for any upper bound y of B.*

(iv) *$a \in A$ is the greatest lower bound of B if a is lower bound of B and $y \leq a$ for any lower bound y of B.*

DEFINITION 4. (Join and Meet). *Let the join of a and b, $a \vee b$, be the least upper bound of $\{a, b\}$. Let the meet of a and b, $a \wedge b$, be the greatest lower bound of $\{a, b\}$.*

DEFINITION 5. (Lattice). *A partially ordered set $\langle A, \leq \rangle$ is a lattice if and only if:*

(i) *For any $a, b \in A$, $a \vee b \in A$.*

(ii) *For any $a, b \in A$, $a \wedge b \in A$.*

DEFINITION 6. (Ortho and Orthocomplemented Lattice). *A lattice $\langle A, \leq \rangle$ is an ortholattice iff some $a \in A$ is the greatest (call it 1) and some $b \in A$ (call it 0) is the least element with respect to $\leq$. A lattice is an orthocomplemeted lattice if and only if there is an operation $*$ on A such that, for any $a, b \in A$:*

(i) $a \vee a^* = 1$

(ii) $a \wedge a^* = 0$

(iii) $a^{**} = a$

(iv) $(a \wedge b)^* = a^* \vee b^*$

DEFINITION 7. (Distributive). *A lattice $\langle A, \leq \rangle$ is a distributive iff, for any $a, b, c \in A$: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.*