FIL2405/4405: Propositional Modal Logic: Language and Proof Theory

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Introduction FIL2405/4405

- Last week, we looked at Propositional Logic.
- This week, we will look at our first modal logics.
- The focus today will be on Propositional Modal Logic (PML).
- We will look at the language and the proof theory of PML.

Recap: Propositional Logic (Language)

Propositional Logic (PL) is logic of propositions, or statements, and how those statements relate to each other in special ways, i.e., how they relate in terms of the logical connectives.

Every logic is given in a language. The language of PL, $\mathcal{L}_P$:

$p, q, r, t, u, p_1, \ldots$	$\sim, \wedge, \vee, \supset, \equiv$	$($, $)$
Sentence letters	Connectives	Punctuation

The *grammatical rules* for constructing well-formed formulae.

Any well-formed formula (*wff*) is either a sentence letter, or recursively constructed via the following rule: $\sim A$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$, and $(A \equiv B)$ are *wffs*, if A and B are *wffs*.

Propositional Logic (Semantics)

The semantics assigns to each *wff* either 1 (true) or 0 (false) using an *interpretation* function v according to the following constraints.

(i) For every sentence letter p, $v(p) = 1$ or $v(p) = 0$ (but not both).

(ii) Complex formulae are assigned values:

- $(\sim^v)$ $v(\sim p) = 1$ iff $v(p) = 0$; and 0 otherwise
- $(\wedge^v)$ $v(p \wedge q) = 1$ iff $v(p) = 1$ and $v(q) = 1$; and 0 otherwise
- $(\vee^v)$ $v(p \vee q) = 1$ iff $v(p) = 1$ or $v(q) = 1$; and 0 otherwise
- $(\supset^v)$ $v(p \supset q) = 1$ iff $v(p) = 0$ or $v(q) = 1$; and 0 otherwise
- $(\equiv^v)$ $v(p \equiv q) = 1$ iff $v(p) = v(q)$; and 0 otherwise

$\Gamma \models A$ iff ... there is no interpretation v which assigns each 1 to formulae in Γ and assigns 0 to A. A is a tautology iff ... $\models A$.

Propositional Logic (Proof Theory)

An axiom is ... a specially selected *wff*.

A transformation rule is ... a rule which licenses various operations on *wffs*, usually theorems.

An Axiom System is ... a way of specifying a logic. Consists of specification of the language of the logic, a set of axioms, and a set of transformation rules.

A theorem is ... a *wff*, resulting from applying transformation rules to axioms or the results of transformations rules applied to axioms.

Modality

So far, we have only be concerned with what is true or false.

But, often we want to make further distinctions. Consider:

(1) London is the capital of the UK.
(2) London is London.

Both (1) and (2) are true. However, (2) is *necessarily* true.

Modality

We can make similar distinctions between claims which are false.

(3) I am 6'10"

(4) I am 6'10" *and* 6'11"

Both (3) and (4) are false. However, (4) is *necessarily* false.

Modality

Generally, modality is the phenomenon of things being so-and-so in a particular way, i.e., the way in which some thing is.

(2) is true and (4) is false, both are the way they are *necessarily*.

(1) is true and (3) is false, however both *could have been otherwise*.

In this course, we will focus on modalities of truth, what are often called *alethic* modalities. However, there are other kinds. For example:

(5)	The Bishop *must* only move along the diagonal.	Rules-based
(6)	You *can't* make the train on time.	Practicality-based

The Need for a new logic

This course is not just about modality, it's about modal *logic*.

Why bother with inventing new logics specifically for modality?

Well, we very often argue using modal notions nd PL is simply not good enough to capture the distinctively modal patterns of reasoning.

Simple Modal Argument

It must be the case that if I am a football fan, then I am a fan of some sport. I'm not a football fan, but I might have been a football fan. Therefore: I might have been a fan of some sport

The Need for a new logic

This simple modal argument cannot be paraphrased into $\mathcal{L}_P$.

$\underbrace{\text{It must be the case that}}_{?}\ \underbrace{\text{if I am a football fan, then I am a fan of some sport}}_{p \supset q}$

$\underbrace{\text{I'm not a football fan}}_{\sim p},\ \underbrace{\text{but I might have been a football fan.}}_{?}$

$\underbrace{\text{I might have been a fan of some sport}}_{?}$

The Need for a new logic

The only way to formalise this simple modal argument is as follows.

- r: It must be the case that if I am a football fan, then I am a fan of some sport.
- s: I'm not a football fan, but I might have been a football fan.
- t: I might have been a fan of some sport.

But an argument of the form $r, s \therefore t$ is clearly invalid.

What we need is a way of expressing the sentential operators 'It must be the case that...' and 'It might be the case that...'

Propositional Modal Logic

The first innovation: we extend the lexicon of $\mathcal{L}_P$ to include two new sentential modal operators, L (for necessity) and M (for possibility).

Lexicon of the Language of Modal Propositional Logic $\mathcal{L}_\rho^M$

The lexicon of $\mathcal{L}_\rho^M$ consists of, for every natural number n:

- Sentence letters $p_n, q_n, r_n, s_n, t_n, u_n$

Logical connectives:

- $\sim$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (material conditional), $\equiv$ (biconditional), and L (necessity)

Punctuation:

- Brackets (, and).

M is not a primitive but is *defined*in terms of L: $M\alpha =_{df} \sim L \sim \alpha$.

Propositional Modal Logic

We then also extend the grammatical rules of $\mathcal{L}_P$ to include L.

Grammar of $\mathcal{L}_P^M$

The well-formed formulae (*wff*) of $\mathcal{L}_P^M$ are all and only those strings of symbols which are either sentence letters or which can be recursively generated from the sentence letters by the following rules:

($\sim$) If A is a *wff*, then $\sim A$ is a *wff*

(L) If A is a *wff*, then LA is a *wff*

($\wedge$) If A and B are *wff*s, then $(A \wedge B)$ is a *wff*

($\vee$) If A and B are *wff*s, then $(A \vee B)$ is a *wff*

($\supset$) If A and B are *wff*s, then $(A \supset B)$ is a *wff*

($\equiv$) If A and B are *wff*s, then $(A \equiv B)$ is a *wff*

Propositional Modal Logic

Consider the Simple Modal Argument. This can now be formalised.

Let $p :=$ I am a football fan and $q :=$ I am a fan of some sport.

(1) $\underbrace{\text{It must be the case that if I am a football fan, then I am a fan of some sport.}}_{L(p \supset q)}$

(2) $\underbrace{\text{I'm not a football fan, but I might have been a football fan.}}_{\sim p \wedge Mp}$

(3) $\underbrace{\text{I might have been a fan of some sport.}}_{Mq}$

Modal Logic K

That's the *language* of Propositional Modal Logic. Now let's look at some logics. This week we focus on Axiom Systems for Modal Logics.

The simplest (and weakest) propositional modal logic is what we call K.

There are two axioms for K:

Axioms for K
(PC) If α is a valid *wff* of PL, then α is an axiom.
(K) $L(p \supset q) \supset (Lp \supset Lq)$

Modal Logic K

There are three rules of inference in K.

Rules of Inference

(US) The result of uniformly replacing any variable or variables $p_1, ..., p_n$ in a theorem by any *wff* $\beta_1, ..., \beta_n$ respectively is itself a theorem.

(MP) If α and $\alpha \supset \beta$ are theorems, so is β.

(N) If α is a theorem, then so is $L\alpha$.

α is a theorem of K ($\vdash_k \alpha$) just in case α can be derived by applying rules of inference to axioms. β follows from Γ in K iff $\Gamma \vdash_k \beta$.

Modal Logic K

$L(p \supset q), Mp \vdash_k Mq$. Proof:

(1)	$L(p \supset q)$	Premise
(2)	Mp	Premise
(3)	$(p \supset q) \supset ({\sim}q \supset {\sim}p)$	(PC)
(4)	$L((p \supset q) \supset ({\sim}q \supset {\sim}p))$	(N)
(5)	$L((p \supset q) \supset ({\sim}q \supset {\sim}p)) \supset (L(p \supset q) \supset L({\sim}q \supset {\sim}p))$	(K)
(6)	$L(p \supset q) \supset L({\sim}q \supset {\sim}p)$	(4), (5), + (MP)
(7)	$L({\sim}q \supset {\sim}p)$	(1), (6), + (MP)
(8)	$L({\sim}q \supset {\sim}p) \supset (L{\sim}q \supset L{\sim}p)$	(K)
(9)	$L{\sim}q \supset L{\sim}p$	(7), (8), + (MP)
(10)	$(L{\sim}q \supset L{\sim}p) \supset ({\sim}L{\sim}p \supset {\sim}L{\sim}q)$	(PC)
(11)	${\sim}L{\sim}p \supset {\sim}L{\sim}q$	(9), (10), (MP)
(12)	$Mp \supset {\sim}L{\sim}p$	Definition of M
(13)	${\sim}L{\sim}p$	(2), (11), (MP)
(14)	${\sim}L{\sim}q$	(11), (13), (MP)
(15)	${\sim}L{\sim}q \supset Mq$	Definition of M
(16)	Mq	(14), (15), (MP)

Modal Logic K

There are several derived rules transformation rules. Here's a useful one.

Substitution of Equivalents (Eq)

If α is a theorem and β is a *wff* which differs from α only in having some *wff* δ at one or more places where α has a *wff* γ, then if $\gamma \equiv \delta$ is a theorem, then β is a theorem.

For instance, $\alpha := (p \supset q)$ and $\beta := (p \vee p) \supset q$, then α and β only differ insofar as where α has p, β has $p \vee p$. By (Eq), then if α is a theorem and $p \equiv (p \vee p)$ is a theorem, then β is a theorem.

Modal Logic K

Let's use (Eq) to prove a nice result. Namely, $\vdash_k Lp \equiv {\sim}M{\sim}p$.

(1)	$\vdash_k Mp \equiv {\sim}L{\sim}p$	Definition of M
(2)	$\vdash_k M{\sim}p \equiv {\sim}L{\sim}{\sim}p$	(1), (US)$[p/{\sim}p]$
(3)	$\vdash_k p \equiv {\sim}{\sim}p$	(PC)
(4)	$\vdash_k M{\sim}p \equiv {\sim}Lp$	(2), (3), (Eq)
(5)	$\vdash_k (M{\sim}p \equiv {\sim}Lp) \supset (Lp \equiv {\sim}M{\sim}p)$	(PC)
(6)	$\vdash_k Lp \equiv {\sim}M{\sim}p$	(4), (5), (MP)

Modal Logic T

K is quite a weak logic, i.e., some statements are not theorems of K, even though they seem to be right about alethic modality. For instance:

(i) $\nvdash_k Lp \supset p$ (It's not a K-theorem that necessarily p then p.)

(ii) $\nvdash_k p \supset Mp$ (It's not a K-theorem that if p then possibly p.)

Both seem right, though! To capture these theorems, we can look at a different system called T. The axioms and inference rules of T are all of the axioms and inference rules of K with the following addition.

(T) $Lp \supset p$ is an axiom.

Modal Logic T

So, T is the system which contains (K), (T), and (PC) as axioms and (US), (MP), and (N) as transformation rules.

Since T has an extra axiom, there are theorems of T which are not theorems of K. We will write that formula α is a theorem of T as $\vdash_t \alpha$.

Since T is just K with an additional axiom, every theorem of K is a theorem of T, i.e., for any *wff* α of $\mathcal{L}_\rho^M$, if $\vdash_k \alpha$, then $\vdash_t \alpha$.

Modal Logic T

$\vdash_t p \supset Mp$.

(1)	$\vdash_t L{\sim}p \supset {\sim}p$	(US), $\beta = {\sim}p$, (T)
(2)	$\vdash_t (L{\sim}p \supset {\sim}p) \supset (p \supset {\sim}L{\sim}p)$	(PC)
(3)	$\vdash_t p \supset {\sim}L{\sim}p$	(1), (2), (MP)
(4)	$\vdash_t p \supset Mp$	Definition M, (3), (Eq)

Modal Logic S4

Even though T extends K, T is still a relatively weak logic.

In particular, T doesn't allow us to prove anything interesting about iterated modal operators. Moreover:

(i) $\nvdash_t Lp \supset LLp$

(It's not a T-theorem that necessarily p, then necessarily necessarily p.)

'$Lp \supset LLp$' is the thesis that necessity holds necessarily.

Next week, we will look at some philosophical arguments for and against this idea. But, for now, we'll just focus on a common logic which builds in $Lp \supset LLp$—the logic called S4.

Modal Logic S4

The axioms and inference rules of S4 are all of the axioms and inference rules of T with the following addition.

(4) $Lp \supset LLp$ is an axiom.

Since S4 has an extra axiom, there are theorems of S4 which are not theorems of T or K. We write that formula α is a theorem of S4 as $\vdash_4 \alpha$.

Since S4 is just T with an additional axiom, every theorem of K or T is a theorem of S4, i.e., for any *wff* α of $\mathcal{L}_\rho^M$, if $\vdash_k \alpha$, then $\vdash_4 \alpha$ and if $\vdash_t \alpha$, then $\vdash_4 \alpha$.

Modal Logic S4

Let's prove $\vdash_4 MMp \supset Mp$.

(1)	$\vdash_4 Lp \supset LLp$	(4)
(2)	$\vdash_4 L{\sim}p \supset LL{\sim}p$	(US), (1)
(3)	$\vdash_4 (L{\sim}p \supset LL{\sim}p) \supset ({\sim}LL{\sim}p \supset {\sim}L{\sim}p)$	(PC)
(4)	$\vdash_4 {\sim}LL{\sim}p \supset {\sim}L{\sim}p$	(2), (3), (MP)
(5)	$\vdash_4 {\sim}L{\sim}p \equiv Mp$	Definition M
(6)	$\vdash_4 {\sim}LL{\sim}p \supset Mp$	(4), (5), (Eq)
(7)	$\vdash_4 (Lp \equiv {\sim}M{\sim}p) \supset ({\sim}Lp \equiv M{\sim}p)$	(PC)
(8)	$\vdash_4 Lp \equiv {\sim}M{\sim}p$	
(9)	$\vdash_4 {\sim}Lp \equiv M{\sim}p$	(7), (8), (MP)
(10)	$\vdash_4 {\sim}LL{\sim}p \equiv M{\sim}L{\sim}p$	(9), (US)$[p/L{\sim}p]$
(11)	$\vdash_4 M{\sim}L{\sim}p \supset Mp$	(6), (10), (Eq)
(12)	$\vdash_4 MMp \supset Mp$	Definition M

Modal Logic S5

S4 includes captures the necessity of necessity, but not the necessity of *possibility*: $\nvdash_4 Mp \supset LMp$

The strongest propositional modal logic we will discuss in this course includes $Mp \supset LMp$ as an axiom. This is the modal logic S5.

The axioms and inference rules of S5, then, are all of the axioms and inference rules of S4 with the following addition.

(5) $Mp \supset LMp$

Again, there are theorems of S5 which are not theorems of S4, nor T, nor K. Again, $\vdash_k \alpha \rightarrow \vdash_t \alpha \rightarrow \vdash_4 \alpha \rightarrow \vdash_5 \alpha$

Modal Logic B

There is one more modal logic to cover. This is B.

B is the result of adding the axiomatic base of T the following axiom.

(B) $p \supset LMp$

S5 can also be defined as the logic which results from adding (5) to B.

Naturally, $\vdash_k \alpha \rightarrow \vdash_t \alpha \rightarrow \vdash_b \alpha$. But: $\vdash_4 \alpha \not\leftrightarrow \vdash_b \alpha$.

All the logics together

The Logic	The Inference Rules	The Axiomatic Base
K	(MP) $\vdash_k \alpha, \vdash_k \alpha \supset \beta \rightarrow \vdash_k \beta$	
	(US) *Uniform Substitution*	
	(N) $\vdash_k \alpha \rightarrow \vdash_k L\alpha$	

All the logics together

The Logic	The Inference Rules	The Axiomatic Base
K	(MP) $\vdash_k \alpha, \vdash_k \alpha \supset \beta \rightarrow \vdash_k \beta$	(PC) Tautologies
	(US) *Uniform Substitution*	(K) $L(p \supset q) \supset (Lp \supset Lq)$
	(N) $\vdash_k \alpha \rightarrow \vdash_k L\alpha$	

All the logics together

The Logic	The Inference Rules	The Axiomatic Base
K	(MP) $\vdash_k \alpha, \vdash_k \alpha \supset \beta \rightarrow \vdash_k \beta$	(PC) Tautologies
	(US) *Uniform Substitution*	(K) $L(p \supset q) \supset (Lp \supset Lq)$
	(N) $\vdash_k \alpha \rightarrow \vdash_k L\alpha$	
T	Same as above.	

All the logics together

The Logic	The Inference Rules	The Axiomatic Base
K	(MP) $\vdash_k \alpha, \vdash_k \alpha \supset \beta \rightarrow \vdash_k \beta$ (US) *Uniform Substitution* (N) $\vdash_k \alpha \rightarrow \vdash_k L\alpha$	(PC) Tautologies (K) $L(p \supset q) \supset (Lp \supset Lq)$
T	Same as above.	(PC), (K), and (T) $Lp \supset p$

All the logics together

The Logic	**The Inference Rules**	**The Axiomatic Base**
K	(MP) $\vdash_k \alpha, \vdash_k \alpha \supset \beta \rightarrow \vdash_k \beta$	(PC) Tautologies
	(US) *Uniform Substitution*	(K) $L(p \supset q) \supset (Lp \supset Lq)$
	(N) $\vdash_k \alpha \rightarrow \vdash_k L\alpha$	
T	Same as above.	(PC), (K), and (T) $Lp \supset p$
S4		

All the logics together

The Logic	The Inference Rules	The Axiomatic Base
K	(MP) $\vdash_k \alpha, \vdash_k \alpha \supset \beta \rightarrow \vdash_k \beta$	(PC) Tautologies
	(US) *Uniform Substitution*	(K) $L(p \supset q) \supset (Lp \supset Lq)$
	(N) $\vdash_k \alpha \rightarrow \vdash_k L\alpha$	
T	Same as above.	(PC), (K), and (T) $Lp \supset p$
S4	Same as above.	

All the logics together

The Logic	The Inference Rules	The Axiomatic Base
K	(MP) $\vdash_k \alpha, \vdash_k \alpha \supset \beta \rightarrow \vdash_k \beta$ (US) *Uniform Substitution* (N) $\vdash_k \alpha \rightarrow \vdash_k L\alpha$	(PC) Tautologies (K) $L(p \supset q) \supset (Lp \supset Lq)$
T	Same as above.	(PC), (K), and (T) $Lp \supset p$
S4	Same as above.	(PC), (K), (T), and (4) $Lp \supset LLp$

All the logics together

The Logic	The Inference Rules	The Axiomatic Base
K	(MP) $\vdash_k \alpha, \vdash_k \alpha \supset \beta \rightarrow \vdash_k \beta$ (US) *Uniform Substitution* (N) $\vdash_k \alpha \rightarrow \vdash_k L\alpha$	(PC) Tautologies (K) $L(p \supset q) \supset (Lp \supset Lq)$
T	Same as above.	(PC), (K), and (T) $Lp \supset p$
S4	Same as above.	(PC), (K), (T), and (4) $Lp \supset LLp$
B		

All the logics together

The Logic	The Inference Rules	The Axiomatic Base
K	(MP) $\vdash_k \alpha, \vdash_k \alpha \supset \beta \rightarrow \vdash_k \beta$ (US) *Uniform Substitution* (N) $\vdash_k \alpha \rightarrow \vdash_k L\alpha$	(PC) Tautologies (K) $L(p \supset q) \supset (Lp \supset Lq)$
T	Same as above.	(PC), (K), and (T) $Lp \supset p$
S4	Same as above.	(PC), (K), (T), and (4) $Lp \supset LLp$
B	Same as above.	

All the logics together

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T	Same as above.	(PC), (K), and (T) $Lp \supset p$
S4	Same as above.	(PC), (K), (T), and (4) $Lp \supset LLp$
B	Same as above.	(PC), (K), (T), and (B) $p \supset LMp$

All the logics together

The Logic	The Inference Rules	The Axiomatic Base
K	(MP) $\vdash_k \alpha, \vdash_k \alpha \supset \beta \rightarrow \vdash_k \beta$ (US) *Uniform Substitution* (N) $\vdash_k \alpha \rightarrow \vdash_k L\alpha$	(PC) Tautologies (K) $L(p \supset q) \supset (Lp \supset Lq)$
T	Same as above.	(PC), (K), and (T) $Lp \supset p$
S4	Same as above.	(PC), (K), (T), and (4) $Lp \supset LLp$
B	Same as above.	(PC), (K), (T), and (B) $p \supset LMp$
S5		

All the logics together

The Logic	The Inference Rules	The Axiomatic Base
K	(MP) $\vdash_k \alpha, \vdash_k \alpha \supset \beta \rightarrow \vdash_k \beta$ (US) *Uniform Substitution* (N) $\vdash_k \alpha \rightarrow \vdash_k L\alpha$	(PC) Tautologies (K) $L(p \supset q) \supset (Lp \supset Lq)$
T	Same as above.	(PC), (K), and (T) $Lp \supset p$
S4	Same as above.	(PC), (K), (T), and (4) $Lp \supset LLp$
B	Same as above.	(PC), (K), (T), and (B) $p \supset LMp$
S5	Same as above.	

All the logics together

The Logic	The Inference Rules	The Axiomatic Base
K	(MP) $\vdash_k \alpha, \vdash_k \alpha \supset \beta \rightarrow \vdash_k \beta$ (US) *Uniform Substitution* (N) $\vdash_k \alpha \rightarrow \vdash_k L\alpha$	(PC) Tautologies (K) $L(p \supset q) \supset (Lp \supset Lq)$
T	Same as above.	(PC), (K), and (T) $Lp \supset p$
S4	Same as above.	(PC), (K), (T), and (4) $Lp \supset LLp$
B	Same as above.	(PC), (K), (T), and (B) $p \supset LMp$
S5	Same as above.	(PC), (K), (T), (4)/(B) (5) $Mp \supset LMp$

Summary

1. We recapped Propositional Logic.
2. We then looked at some motivation for developing modal logic.
3. We looked at how to specify the language of Modal Propositional Logic.
4. We then looked at axiom systems for five common modal logics: K, T, S4, B, and S5.
5. We also looked at how we should approach proofs in these systems.