FIL2405/4405: Propositional Modal Logic: Semantics

10th February

Introduction & plan for today

This week will look at a common semantics for PML. This is a model-theoretic semantics, often known as possible worlds semantics.

We look at how to semantically understand the systems K, T, S4, B, S5.

We will then look at the relationship between the semantic and syntactic characterisation of these logics, particularly completeness.

Recap

The language of PML, $\mathcal{L}_\rho^M$, extends $\mathcal{L}_\rho^M$ with an operator L.

If p is a *wff* of $\mathcal{L}_\rho^M$, then Lp is too. We define $Mp =_{df} \sim L \sim p$.

The weakest modal logic we consider is K. The axiomatic basis of K is:

(PC) If α is a valid *wff* of PL, then α is an axiom.
(K) $L(p \supset q) \supset (Lp \supset Lq)$

The transformation rules for K are the following three.

(MP) If α and $\alpha \supset \beta$ are theorems, then β is a theorem.
(N) If α is a theorem, then $L\alpha$ is a theorem.
(US) The result of uniformly replacing variables $p_1, \ldots, p_n$ in a theorem with *wff* $\beta_1, \ldots, \beta_n$ is itself a theorem.

Stronger Logics

All other logics considered in this course are stronger than K.

All other logics considered have the same transformation rules.

Each has a stronger axiomatic base, adding axioms to the axioms of K.

For the logic T, we add: (T) $Lp \supset p$

For the logic B, we add (T) $Lp \supset p$ and (B) $p \supset LMp$

For the logic S4, we add (T) $Lp \supset p$ and (4) $Lp \supset LLp$

For the logic S5, we add (T) $Lp \supset p$ and (5) $Mp \supset LMp$

Introduction to Possible World (PW) Semantics

For $\mathcal{L}_\rho$ we used valuations v for the semantics. Valuations assigned truth values to formulae. For $\mathcal{L}_\rho^M$ we have to complicate this.

Consider: under what conditions is 'Lp', or 'Necessarily, p' true?

'Necessarily, p' is true *iff* 'p' is not possibly false
iff there's no way in which 'p' is false
iff 'p' is true in every possible world!

Possible worlds are intuitively 'total ways the world could have been'.

Introduction to Possible World (PW) Semantics

We want to assign truth values to all formulae in a way which also allows us to assign truth values to modal formulae.

The crucial idea: we assign truth values to formulae of $\mathcal{L}_\rho^M$ relative to possible worlds. We write $v(\alpha, w) = 1$ for α is true relative to w.

This allows us to capture the intuitive idea.

(L^v) $v(Lp, w) = 1$ iff for every possible world w': $v(p, w') = 1$.
(M^v) $v(Mp, w) = 1$ iff for some possible world w': $v(p, w') = 1$.

Introduction to Possible World (PW) Semantics

We want a semantics for $\mathcal{L}_{\rho}^{M}$ to study modal logic.

For PL, the valid formulae were true under *any* interpretation v.

Now formulae are assigned truth values relative to possible worlds.

So, we now need to generalise over worlds, as well as assignments. To do this, we make use of frames and models based on frames.

A Frame

A *frame* $\mathfrak{F} = \langle W, R \rangle$ is an ordered pair, where W is a non-empty set (of 'worlds') and R is a binary relation on W, i.e., for any members $w, w' \in W$, it is determinate whether Rww' or $\sim Rww'$.

Introduction to Possible World (PW) Semantics

A Model

A *model* $\mathfrak{M} = \langle W, R, v \rangle$ is an ordered triple, where $\langle W, R \rangle$ is a frame and v is a valuation function. Note, we say that the model $\langle W, R, v \rangle$ is *based on* the frame $\langle W, R \rangle$...

Pause: what's R? R is often called an accessibility relation.

Intuitively, think of Rww' as saying that w' is possible relative to w.

Consider:

w_1 : I am in Oslo at t and technology is as it actually is

w_2 : I am in London at $t + 5$mins

If we are interested in some sort of practical possibility, $\sim Rw_1w_2$.

Introduction to Possible World (PW) Semantics

A Model

A *model* $\mathfrak{M} = \langle W, R, v \rangle$ is an ordered triple, where $\langle W, R \rangle$ is a frame and v is a valuation function. Note, we say that the model $\langle W, R, v \rangle$ is *based on* the frame $\langle W, R \rangle$. v satisfies, for *wff* α, β and $w \in W$:

($\sim^v$) $v(\sim\alpha, w) = 1$ iff $v(\alpha, w) = 0$; 0 otherwise.
($\wedge^v$) $v(\alpha \wedge \beta, w) = 1$ iff $v(\alpha, w) = 1$ and $v(\beta, w) = 1$; 0 otherwise.
($\vee^v$) $v(\alpha \vee \beta, w) = 1$ iff $v(\alpha, w) = 1$ or $v(\beta, w) = 1$; 0 otherwise.
($\supset^v$) $v(\alpha \supset \beta, w) = 1$ iff $v(\alpha, w) = 0$ or $v(\beta, w) = 1$; 0 otherwise.
($\equiv^v$) $v(\alpha \equiv \beta, w) = 1$ iff $v(\alpha, w) = v(\beta, w)$. 0 otherwise.
(L^v) $v(L\alpha, w) = 1$ iff for every $w' \in W$: $v(\alpha, w') = 1$; 0 otherwise
(M^v) $v(M\alpha, w) = 1$ iff for some $w' \in W$: $v(\alpha, w') = 1$; 0 otherwise

Introduction to Possible World (PW) Semantics

We can now define some useful semantic notions.

We write $\mathfrak{M}, w \vDash p$ when p is true relative to a world w in model.

We write $\mathfrak{M} \vDash p$ when p is true relative to every world in $\mathfrak{M}$. We will often say that p is valid in the model $\mathfrak{M}$ in this case.

We say that p is true in a frame $\mathfrak{F}$ if $\mathfrak{M} \vDash p$, for every $\mathfrak{M}$ based on $\mathfrak{F}$. We will often say that p is valid in the frame $\mathfrak{F}$ in this case.

We say that p is valid if it is valid in every frame $\mathfrak{F}$.

Examples

$\mathfrak{M}$ is $\langle W, R, v\rangle$, where $W = \{w_1, w_2, w_3\}$, $R : Rw_1w_2, Rw_2w_3, Rw_3w_1$, and $v(p, w_1) = v(p, w_2) = 1$, and $v(p, w_3) = 0$.

$\mathfrak{M} \vDash Mp$?	No: $\mathfrak{M}, w_3 \nvDash p$ and Rw_2w_3.
$\mathfrak{M} \vDash Mp \supset p$?	No: $\mathfrak{M}, w_3 \vDash \Diamond p \wedge {\sim}p$.
$\mathfrak{M} \vDash {\sim}p \supset Mp$?	Yes. If $\mathfrak{M}, w \vDash {\sim}p$, then $w = w_3$. $\mathfrak{M}, w_3 \vDash Mp$

Logic K

How does this semantics relate to the systems we know?

Definition of K-validity

Let a *wff* α is K-valid iff α is valid in all frames $\mathfrak{F}$.

K-validity Theorem (Soundness)

If $\vdash_k \alpha$ (α is a theorem of K), then α is K-valid.

Proof Sketch. We show that all the axioms of K are K-valid and all the transformation rules are K-validity preserving. If α is a theorem of K (result of applying the transformation rules to axioms), α is K-valid.

Logic T and T-Validity

We get similar results to K-validity Theorem for other systems by restricting the class of all frames $\mathfrak{F}$ in terms of constraints on R.

Consider T. T's extra axiom: (T) $Lp \supset p$.

$Lp \supset p$ is not K-valid, i.e., not valid in any frame $\mathfrak{F}$.

Proof. $Lp \supset p$ is not K-valid iff $Lp \supset p$ is not valid in some frame $\langle W, R \rangle$ iff there is a model $\mathfrak{M} = \langle W, R, v \rangle$ based on some $\langle W, R \rangle$ in which $Lp \supset p$ fails to hold at some $w \in W$. Let $W = \{w_1, w_2\}$, $R : R12$, and $v(p, w_1) = 0$ and $v(p, w_2) = 1$. $\mathfrak{M}, w_1 \vDash Lp \wedge \sim p$.

Logic T and T-Validity

We define a class of frames in which all theorems of T are valid by defining the class of frames in which R is reflexive.

T-frame and T-validity

Let a T-frame be a frame $\langle W, R\rangle$, where R is a reflexive relation, i.e., for every $w \in W$: Rww. A *wff* α is T-valid iff α is valid in every T-frame.

T-validity Theorem

If $\vdash_t \alpha$ (α is a theorem of T), then α is T-valid.

Proof. Suppose $\langle W, R, v\rangle$ is an arbitrary model $\mathfrak{M}$ based on an arbitrary T-frame $\langle W, R\rangle$. Suppose $\mathfrak{M}, w \vDash Lp$, for arbitrary $w \in W$. $\langle W, R\rangle$ is a T-frame, so R is reflexive, so Rww. Therefore: $\mathfrak{M}, w \vDash p$.

S4-Validity

Next consider S4. S4's extra axiom: (4) $Lp \supset LLp$.

$Lp \supset LLp$ is not T-valid, i.e, not valid in any reflexive frame $\mathfrak{F}$.

Proof. Let $\mathfrak{M} = \{W, R, v\}$, where $W = \{w_1, w_2, w_3\}$, R is reflexive, Rw_1w_2, and Rw_2w_3, and $v(p, w_1) = v(p, w_2) = 1$ and $v(p, w_3) = 0$. $\mathfrak{M}, w_1 \vDash Lp$, since for every w' such that Rww': $\mathfrak{M}, w' \vDash p$. But, $\mathfrak{M}, w_1 \nvDash LLp$, since $\mathfrak{M}, w_2 \nvDash Lp$ (because Rw_2w_3 and $v(p, w_3) = 0$ and so $\mathfrak{M}, w_3 \nvDash p$) and Rw_1w_2. So, $\mathfrak{M}, w_1 \nvDash Lp \supset LLp$.

Logic S4 and 4-Validity

We define a class of frames in which all theorems of S4 are valid by defining the class of frames in which R is reflexive and transitive.

Relation R is transitive iff, for every x, y, z: if Rxy and Ryz, then Rxz.

S4-frame and S4-validity

Let an S4-frame be a frame $\langle W, R \rangle$, where R is reflexive and transitive. A *wff* α is S4-valid iff α is valid in every S4-frame.

S4-validity Theorem

If $\vdash_4 \alpha$ (α is a theorem of S4), then α is S4-valid.

Logic S4 and S4-Validity

Here is a proof by *reductio* of S4-validity Theorem.

Proof. Suppose $\langle W, R, v\rangle$ is an arbitrary model $\mathfrak{M}$ based on a S4-frame.
Suppose as well that $\mathfrak{M}, w \vDash Lp \wedge {\sim}LLp$, for arbitrary $w \in W$.
If $\mathfrak{M}, w \vDash Lp \wedge {\sim}LLp$, then $\mathfrak{M}, w \vDash Lp$.
If $\mathfrak{M}, w \vDash {\sim}LLp$, then some w': Rww': $\mathfrak{M}, w' \nvDash Lp$.
If $\mathfrak{M}, w' \nvDash Lp$, then some w'': $Rw'w''$: $\mathfrak{M}, w'' \nvDash p$.
If Rww' and $Rw'w''$ and R is transitive, then Rww''.
Since $\mathfrak{M}, w'' \nvDash p$ and Rww'', $\mathfrak{M}, w \nvDash Lp$.
Contradiction ($\mathfrak{M}, w \vDash Lp$ and $\mathfrak{M}, w \nvDash Lp$)!
Therefore, for arbitrary $\mathfrak{M}$ based on a S4-frame: $\mathfrak{M} \vDash Lp \supset LLp$

Logic B and B-Validity

Next consider B. B's extra axiom: (B) $p \supset LMp$.

$p \supset LMp$ is not T-valid, K-valid, or S4-valid.

Proof. Let $\mathfrak{M} = \{W, R, v\}$, where $W = \{w_1, w_2, w_3\}$, where R is reflexive, transitive, and where Rw_1w_2 but $\sim Rw_2w_1$. Suppose, as well, that $v(p, w_1) = 1$ and $v(p, w_2) = v(p, w_3) = 0$. To begin, $\mathfrak{M}, w_1 \vDash p$. Moreover, $M, w_2 \vDash \sim Mp$, since if Rw_2w', then $w' = w_2$. Therefore, $\mathfrak{M}, w_1 \vDash \sim LMp$. Thus, $\mathfrak{M}, w_1 \nvDash p \supset LMp$.

Logic B and B-Validity

We define a class of frames in which all theorems of B are valid by defining the class of frames in which R is reflexive and symmetric.

Relation R is symmetric iff, for every x, y: if Rxy, then Ryx.

B-frame and B-validity

Let a B-frame be a frame $\langle W, R \rangle$, where R is reflexive and symmetric. A *wff* α is B-valid iff α is valid in every B-frame.

B-validity Theorem

If $\vdash_b \alpha$ (α is a theorem of B), then α is B-valid.

Logic B and B-Validity

Here is a proof of the B-validity Theorem.

Proof. Suppose $\langle W, R, v\rangle$ is an arbitrary model $\mathfrak{M}$ based on a symmetric frame. Suppose $\mathfrak{M}, w \vDash p$. Consider any w' such that Rww'.Since R is symmetric, if Rww', then $Rw'w$.
Since $\mathfrak{M}, w \vDash p$ and $Rw'w$, $\mathfrak{M}, w' \vDash Mp$.
Given that w' was any $w \in W$ such that Rww', $\mathfrak{M}, w \vDash LMp$.

Logic S5 and S5-Validity

Finally, consider S5. S5's extra axiom: (5) $Mp \supset LMp$

$Mp \supset LMp$ is not T-valid, K-valid, S4-valid, or B-valid.

(Proof of this is an exercise for you!)

We define a class of frames in which all theorems of S5 are valid by defining the class of frames in which R is an equivalence relation.

R is an equivalence relation iff R is reflexive, transitive, and symmetric.

Logic S5 and S5-Validity

S5-frame and S5-validity

Let a S5-frame be a frame $\langle W, R\rangle$, where R is an equivalence relation. A *wff* α is S5-valid iff α is valid in every S5-frame.

S5-Validity Theorem

If $\vdash_5 \alpha$ (α is a theorem of S5), then α is S5-valid.

Completeness

We now have a variety of soundness results:

(K) If $\vdash_k \alpha$ (α is a theorem of K), then α is K-valid.
(Valid in *all* frames.)

(T) If $\vdash_t \alpha$ (α is a theorem of T), then α is T-valid.
(Valid in *all* reflexive frames.)

(S4) If $\vdash_4 \alpha$ (α is a theorem of S3), then α is S4-valid.
(Valid in *all* reflexive and transitive frames.)

(B) If $\vdash_b \alpha$ (α is a theorem of B), then α is B-valid.
(Valid in *all* reflexive and symmetric frames.)

(S5) If $\vdash_5 \alpha$ (α is a theorem of S5), then α is S5-valid.
(Valid in *all* reflexive, transitive, and symmetric frames.)

Completeness

However, this does not guarantee the converse.

For instance, the K-validity Theorem does not guarantee that all K-validities are theorems of K.

For this, we need a completeness result, e.g., we need to show that if some *wff* α is K-valid, then α is a theorem of K.

In what follows, we will show a general completeness result for any consistent normal modal propositional logic.

(*A normal modal logic is, for our purposes, any propositional modal logic which is an extension of K.*)

Canonical Models Proof of Completeness

To prove completeness, we define and prove some results about canonical models. Here's the broad-strokes outline:

> We show that for every normal modal system *S* there is a model, a canonical model, which has a special property: any *wff* α is valid in the canonical model for *S* iff it is a theorem of *S*.

Our starting point is defining maximally consistent sets of *wff* and proving some results about them. Why? Because maximally consistent sets of *wff*s are going to be the worlds in the canonical model.

Maximally S-Consistent Sets of *wff*

S-Consistent Sets of *wff*s

A set of *wff*s Γ is S-consistent set Γ iff no finite collection $\alpha_1, ..., \alpha_n \in \Gamma$ is such that $\vdash_s \sim(\alpha_1 \wedge, ..., \wedge \alpha_n)$.

Maximal Sets of *wff*s

A set of *wff*s Γ is maximal iff for every *wff* α, either $\alpha \in \Gamma$ or $\sim\alpha \in \Gamma$.

Maximally S-consistent Sets of *wff*s

A set Γ is maximally S-consistent iff Γ is maximal and S-consistent.

Maximally S-Consistent Sets of *wff* s

Now some useful results about maximally S-consistent sets of *wff* s.

Lemma 1

Suppose that Γ is a maximally S-consistent set of *wff*. Then:

(i) For any *wff* α, exactly one member of $\{\alpha, \sim\alpha\}$ is in Γ.

(ii) $\alpha \vee \beta \in \Gamma$ iff either $\alpha \in \Gamma$ iff either $\alpha \in \Gamma$ or $\beta \in \Gamma$.

(iii) $\alpha \wedge \beta \in \Gamma$ iff $\alpha \in \Gamma$ and $\beta \in \Gamma$.

(iv) if $\alpha \in \Gamma$ and $\alpha \supset \beta \in \Gamma$, then $\beta \in \Gamma$.

Lemma 2

Suppose that Γ is any maximally S-consistent set of *wff*. Then:

(i) If $\vdash_s \alpha$, then $\alpha \in \Gamma$.

(ii) If $\alpha \in \Gamma$ and $\vdash_s \alpha \supset \beta$ then $\beta \in \Gamma$.

Maximally S-Consistent Sets of *wff* s

Theorem 3

Suppose that Λ is an S-consistent set of *wff*. There is a maximal S-consistent set of *wff* Γ such that $\Lambda \subseteq \Gamma$.

Proof. Order all *wff* of $\mathcal{L}_{\rho}^{M}$, i.e., $\alpha_1, \alpha_2, \ldots$. Then define a sequence $\Gamma_0, \Gamma_1, \ldots$, of sets of *wff* s as follows.

(1) $\Gamma_0 = \Lambda$

(2) Given Γ_n, let Γ_{n+1} be $\Gamma_n \cup \{\alpha_{n+1}\}$ if this is S-consistent and let Γ_{n+1} be $\Gamma_n \cup \{\sim \alpha_{n+1}\}$ if otherwise.

Each Γ_n is S-consistent. Let $\Gamma = \bigcup_{i=0}^{n} \Gamma_i$. Γ is maximally S-consistent.

Modal Features of Maximally S-Consistent Sets

We use these sets of *wff* s as *worlds* when we construct the canonical model. We need to define when such sets access each other by R.

Accessibility R

In the canonical model, $R\Gamma\Delta$ iff for every *wff* β, if $L\beta \in \Gamma$, then $\beta \in \Delta$.

Useful notation: If Λ is a set of *wff*, then let $\mathrm{L}^{-}(\Lambda) = \{\beta : L\beta \in \Lambda\}$.

(Basically, the set of 'necessitated formulae' in Λ.)

Modal Features of Maximally S-Consistent Sets

Question: will R as we have defined it work as we want?

(i) If $Lp \in \Gamma$ and $R\Gamma\Delta$, will $p \in \Delta$?
Yes, because of the definitions of R.

(ii) If $\sim Lp \in \Gamma$, will there be a Δ such that $R\Gamma\Delta$ and $p \in \Delta$?
Yes, but we have to prove that!

Lemma 4

Let S be any normal system of propositional modal logic, and let Λ be an S-consistent set and $\sim L\alpha \in \Lambda$. Then $\mathrm{L}^{-}(\Gamma) \cup \{\sim\alpha\}$ is S-consistent.

Proof of Lemma 4

Suppose: Λ is a maximally S-consistent set of *wff* s.

Suppose: $\sim L\alpha \in \Lambda$ and *yet* $\mathrm{L}^{-}(\Lambda) \cup \{\sim\alpha\}$ is not S-consistent.

If $\mathrm{L}^{-}(\Lambda) \cup \{\sim\alpha\}$ is S-inconsistent, then some finite $\beta_1, ..., \beta_n$ in $\mathrm{L}^{-}(\Lambda)$:

$$\vdash_s \sim(\beta_1 \wedge, ..., \wedge \beta_n \wedge \sim\alpha)$$

So: $\vdash_s (\beta_1 \wedge, ..., \wedge \beta_n) \supset \alpha$. In any normal modal system:

$$\vdash_s L(\beta_1 \wedge, ..., \wedge \beta_n) \supset L\alpha$$

L distributes over conjunction: $\vdash_s (L\beta_1 \wedge, ..., \wedge L\beta_n) \supset L\alpha$.

Thus: $\vdash_s \sim(L\beta_1 \wedge, ..., \wedge L\beta_n \wedge \sim L\alpha)$. Λ is not S-consistent, then!

Canonical Models

This has been leading up to the construction of Canonical Models.

Canonical Models for S

A canonical model for S is a triple $\langle W, R, v\rangle$ such that:

- W: W is the set of all maximally S-consistent set of *wff*s.
- R: For any w, w': Rww' iff for every *wff* β if $L\beta \in w$, then $\beta \in w'$.
  (Alternatively: Rww' iff $\mathrm{L}^{-}(w) \subset w'$)
- v: $v(p, w) = 1$ iff $p \in w$.

Canonical Models

Theorem 5

Let $\langle W, R, v \rangle$ be the canonical model for a normal propositional modal system S. Then for any *wff* α and any $w \in W$, $v(\alpha, w) = 1$ iff $\alpha \in W$.

Proof. We prove by induction on the complexity of formulae. The theorem holds for propositional variables. So, we show that:

(a) If theorem holds for α, then it holds for $\sim\alpha$

(b) If theorem holds for α and β, then it holds for $\alpha \vee \beta$

(c) If theorem holds for α, then it holds for $L\alpha$

Corollary 6

Any *wff* α is valid in the canonical model of S iff $\vdash_s \alpha$.

Completeness

Corollary 6 is general for any normal modal system S.

For completeness of modal system S we discussed earlier, we show that the canonical model of S is in the important class of models for S.

(K) Show the canonical model for K is a model.

(T) Show the canonical model for T is a reflexive model

(S4) Show the canonical model for S4 is a reflexive and transitive model

(B) Show the canonical model for B is a reflexive and symmetric model

(S5) Show the canonical model for S5 is an equivalence model

Summary

We've look at possible worlds semantics for modal logic.

We looked at how to set up sound semantics for K, T, S4, B, and S5.

We looked the completeness of this semanticsfor K, T, S4, B, and S5.