Quantified Modal Logic and Identity

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1. Identity Introduction

Two entities a and b are identical if a and b are the same.[1] Distinguish:

QUALITATIVE: a and b are *qualitatively identical* if a and b share properties.

NUMERICAL: a and b are *numerically identical* if they are one and the same entity.

- There are **many** qualitative identity relations
- Qualitative identity is **gradable**.
- Numerical identity is **singular**, it is **not gradable**.

2. Modal Properties of Identity

Relations of qualitative identity are not necessary, e.g., two cats may be qualitatively identical in several respects, but they did not have to be.

With relations of numerical identity, things are more complicated. Consider:

(1) Eric Blair is George Orwell

Since 'Eric Blair' and 'George Orwell' pick out the very same man, (1) is true.

Moreover, *how could it fail to be true*? (1), if true, is necessarily true.

This suggests that numerical identity statements, unlike qualitative identity statements, are necessarily true, if they are true at all. But we should be careful.

(2) The author of *Nineteen Eighty-Four* is the author of *Coming up for Air*

(2) is true. But it is not necessarily true, since 'The author of *Nineteen Eighty-Four*' and 'the author of *Coming up for Air*' do not necessarily pick out the same person.

We can get clearer on the modal properties of identity statements, then, with Saul Kripke's distinction between *rigid* and *accidental* designators (*NN*: 48).

> Let's call something a *rigid designator* if in every possible world it designates the same object, a *nonrigid* or *accidental designator* if this is not the case.[2,3]

NECESSARY IDENTITY: If 'a' and 'b' are rigid, then $a = b$ is necessarily true, if true.

[1] Here, identity is a relation between entities, not names, see *Naming and Necessity* (*NN*), pp. 107–108.

[2] Note that if 'n' rigidly designates n, this does not mean that 'n' is used by people in all worlds to designate n, but is used *by us* to rigidly designate n.

[3] Kripke also thinks that all names are rigid.

3. More Identity Relations

We often make identity claims which do not concern objects.

(3) To be big and blue is to be blue and big (PROPERTY)

(The property of being big and blue *is identical* to the property of being blue and big.)

(4) Water is H_2O (THEORETICAL)[4]

(To be water is to have the chemical structure H_2O)

[4] For more discussion of this kind of claim, see Putnam *The Meaning of "Meaning"* (1975).

Kripke discusses claims like (4). He thinks that theoretical identifications like (4) are necessarily true, if true much in the same way that (1) is necessarily true.

> It seems to me that any case which someone will think of, which he thinks at first is a case in which heat - contrary to what is actually the case - would have been something other than molecular motion, would actually be a case in which some creatures with different nerve endings from ours inhabit this planet? ... and in which these creatures were sensitive to that something else, say light, in such a way that they felt the same thing that we feel when we feel heat. But this is not a situation in which, say, light would have been heat, or even in which a stream of photons would have been heat, but a situation in which a stream of photons would have produced the characteristic sensations which we call "sensations of heat" (*NN: 131–2*)[5]

[5] This is a common Kripkean strategy, see Arif Ahmed's *Saul Kripke* (2007), pp. 72–81 for a discussion of "Modal Illusions".

For Kripke, natural kind terms and singular terms behave similarly. The key idea:

> The references of 'R_1' and 'R_2', respectively, may well be fixed by nonrigid designators 'D_1' and 'D_1', in the Hesperus and Phosphorus cases these have the form 'the heavenly body in such-and-such position in the sky in the evening (morning)'. Then although $R_1 = R_2$' is necessary, $D_1 = D_2$ may well be contingent, and this is often what leads to the erroneous view that '$R_1 = R_2$' might have turned out otherwise. (*NN*: 141)

If theoretical identifications are indeed necessary, then there are a wide class of *a posteriori* necessities. We empirically discover modal truths.

4. Quantified Modal Logic with Identity[6]

[6] This closely follows *H&C*, pp. 312–314.

LANGUAGE $\mathcal{L}^M_=$. The language for quantified modal logic with identity $\mathcal{L}^M_=$ is an extension of $\mathcal{L}^M_\forall$. We add a two-place logical predicate $=$ to the lexicon.[7] It follows from the grammatical rules governing predicates in $\mathcal{L}^M_\forall$ that:

[7] $\ulcorner = \urcorner$ is a *logical* predicate insofar as it has a fixed meaning across models in the semantics.

(=) If $\ulcorner x \urcorner$ and $\ulcorner y \urcorner$ are variables, then $\ulcorner x = y \urcorner$ is a well-formed formula of $\mathcal{L}^M_=$.

TWO AXIOMS FOR $=$. There are two axioms which govern the behaviour of $=$.

(I1) $x = x$

If α and β differ only in that α has free x in zero or more places where β has free y:

(I2) $x = y \supset (\alpha \supset \beta)$

MODELS WITH $=$. We extend our models $\langle W, R, D, v\rangle$ to handle $=$ by extending v:

- $v(=)$ is the set of triples $\langle u, u, w\rangle$, for every $u \in D$ and $w \in W$.
- $\mathfrak{M}, w, \mu \vDash x = y$ if and only if $\langle \mu(x), \mu(y), w\rangle \in v(=)$.

Straightforwardly, (I1) and (I2) are valid in a semantics with these models.

- $\mu(x) = \mu(x)$, for any assignment μ. Thus, for an arbitrary $\mathfrak{M}, w \in W$ and assignment μ: $\mathfrak{M}, w, \mu \vDash x = x$. Thus, $\mathfrak{M} \vDash x = x$. (I1) is valid.
- Suppose $\mu(x) = \mu(y)$ and suppose for arbitrary $\mathfrak{M}$ and $w \in W$: $\mathfrak{M}, w, \mu \vDash \alpha$. Thus, $\mathfrak{M}, w, \mu \vDash \beta$. Thus, for arbitrary $\mathfrak{M}$: $\mathfrak{M} \vDash x = y \supset (\alpha \supset \beta)$.

In these models, the necessity of identity holds. This is good, since we can *prove* the necessity of identity in LPC + S + (I1)–(I2).

L1 $\vdash x = y \supset Lx = y$

(1)	$\vdash x = y \supset (Lx = x \supset Lx = y)$	(I2)
(2)	$\vdash Lx = x \supset (x = y \supset Lx = y)$	(1) + (PC).[8]
(3)	$\vdash Lx = x$	(I1) + (N)
(4)	$\vdash x = y \supset Lx = y$	(3) + (MP).

[8] Appeal to tautology: $(p \supset (q \supset r)) \supset (q \supset (p \supset r))$

Moreover, in these models, the necessity of *distinctness* holds. If $\mu(x) \neq \mu(y)$, then for any $\mathfrak{M}$ and any $w \in W$: $\mathfrak{M}, w, \mu \vDash \sim(x = y)$. Thus, $\mathfrak{M} \vDash \sim x = y \supset L\sim x = y$.

However, the necessity of distinctness is not provable in systems weaker than B.

In K, we can only derive $M\sim x = y \supset \sim x = y$ and thus $LM\sim x = y \supset L\sim x = y$.

To derive the necessity of distinctness, we need an instance of the B axiom, i.e.,

$$\sim x = y \supset LM\sim x = y$$

For a sound and complete pair involving a semantics defined using the models above and systems of quantified modal logic with identity, we have to also stipulate that the necessity of distinctness holds in the system.

That is, LPC + S + $=$, where S can be either K, T, S4, B, or S5, is the system of LPC + S + (I1)–(I2) *and* the following:

$$\textbf{(LNI)} \quad \sim x = y \supset L\sim x = y$$