FIL2405/4405: Propositional Logic & Basic Logical Notions

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Arguments

One purpose of logic is evaluating arguments.

An argument is a series of sentences. For example:

(1) If Bob is nice, then Bob is cool.
(2) Bob is nice.
$\therefore$ (3) Bob is cool.

(1) and (2) are premises and (3) is the conclusion .

The conclusion (3) is supposed to follow from (1) and (2).

Good Arguments

Intuitively, a good argument is one in which the conclusion really does follow from the premises. We call this a valid argument.

Consider (1)–(3) again. If (1)–(2) are true, then (3) must be true.

Intuitively, the argument (1)–(3) is a valid argument.

Consequence

A sentence C is a consequence of sentences $A_1, ..., A_n$ just in case there is no case in which $A_1, ..., A_n$ are true and yet C is not true.

Valid/Invalid

An argument is valid just in case the conclusion is a consequence of the premises. An argument is invalid if it is not valid.

Some Refinements

First, when we talk about 'sentences', we mean declarative sentences.

(1) The cat is sat on the mat. (Declarative)
(2) Sit on the mat, cat! (Imperative)
(3) The cat is sat on the mat? (Questions)

Second, we should distinguish different kinds of consequence.

Here, we are interested in logical consequence. Logical consequence does not depend on the content of the sentences involved.

An argument is formally valid, if at all, regardless of what the sentences making up the argument mean.

Some Refinements

Contrast the following two arguments.

(1) Sally is a cat
$\therefore$ (2) Sally is a mammal

(1′) Sally is a cat
(2′) If Sally is a cat, then Sally is a mammal
$\therefore$ (3′) Sally is a mammal

The validity of (1)–(2) depends on what ‘cat’ and ‘mammal’ mean.

Some Refinements

Contrast the following two arguments.

(1) Sally is a cat — Valid
$\therefore$ (2) Sally is a mammal — Not Formally Valid

(1′) Sally is a cat
(2′) If Sally is a cat, then Sally is a mammal
$\therefore$ (3′) Sally is a mammal

Validity of (1′)–(3′) doesn't depend on what 'cat' and 'mammal' mean.

Good Arguments Refined

(1′) Sally is a cat
(2′) If Sally is a cat, then Sally is a mammal
∴ (3′) Sally is a mammal

Valid
Formally Valid

Importantly, (1′)–(3′) has a valid form, hence ‘formally’ valid.

(1′) X
(2′) If X, then Y
∴ (3′) Y

Propositional Logic

Logicians and Philosophers are interested in developing tools to precisely and systematically chart patterns of inference.

(1′)–(3′) *looks* good. But why is it? What about other arguments?

One way of answering these questions is to use Propositional Logic.

(This is also known as ‘Propositional Calculus’, ‘Sentential Calculus’, or ‘Sentential Logic’!)

The (rough) idea: we replace sentences with ‘propositional variables’ (p, q, r, etc.) and we replace logical expressions like ‘and’, ‘not’, or ‘if’ with precisely defined logical connectives ($\wedge, \sim, \supset$).

Propositional Logic: The Basics & Language

To start we specify the language $\mathcal{L}_\rho$ of Propositional Logic.

To specify $\mathcal{L}_\rho$ we specify the lexicon and the grammar.

Lexicon of $\mathcal{L}_\rho$

The lexicon of $\mathcal{L}_\rho$ consists of, for every natural number n:

- Sentence letters $p_n, q_n, r_n, s_n, t_n, u_n$

Logical connectives:

- $\sim$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (material conditional), $\equiv$ (biconditional)

Punctuation:

- Brackets (, and).

Propositional Logic: The Basics & Language

Next, we specify the grammar of $\mathcal{L}_\rho$.

The grammar of a language like $\mathcal{L}_\rho$ specifies the *well-formed formulae*.

Grammar of $\mathcal{L}_\rho$

The well-formed formulae (*wff*) of $\mathcal{L}_\rho$ are all and only those strings of symbols which are either sentence letters or which can be recursively generated from the sentence letters by the following rules:

($\sim$) If A is a *wff*, then $\sim A$ is a *wff*

($\wedge$) If A and B are *wff*s, then $(A \wedge B)$ is a *wff*

($\vee$) If A and B are *wff*s, then $(A \vee B)$ is a *wff*

($\supset$) If A and B are *wff*s, then $(A \supset B)$ is a *wff*

($\equiv$) If A and B are *wff*s, then $(A \equiv B)$ is a *wff*

Propositional Logic: The Basics & Language

Is '$p\sim \supset qr)$' a *wff*? × '$((p \supset \sim q) \supset r)$'? ✓ '$p \supset r \supset q$'? ×

Note: often we write '$((p \supset \sim q) \supset r)$' simply as '$(p \supset \sim q) \supset r$'

Pronunciation. Let $p =$ 'Sally is a cat' and $q =$ 'Sally is a mammal'

$\sim p =$ It is not the case that Sally is a cat

$p \wedge q =$ Sally is a cat and Sally is a mammal

$p \vee q =$ Sally is a cat or Sally is a mammal

$p \supset q =$ If Sally is a cat, then Sally is a mammal

$p \equiv q =$ Sally is a cat if and only if Sally is a mammal

Propositional Logic: Semantics

That's the *language*, but what does it *mean*?

To answer this, we should specify a semantics. $\mathcal{L}_\rho$ is a simple language. The semantic value of any formula is simply a truth value.

Things are not so simple, however. Complex formulae should be given a truth-value which is determined by the atomic (sentence letter) parts and the connectives involved.

We'll look at two kinds of semantics for $\mathcal{L}_\rho$: Truth-table, and interpretation semantics for $\mathcal{L}_\rho$.

Truth Tables

A truth table is a way of specifying the meaning of $\sim, \wedge, \vee, \supset, \equiv$.

Each connective is a truth-functional connective: the truth value of a *wff* involving any $\sim, \wedge, \vee, \supset, \equiv$ is determined by the truth value of the atomic (sentence letter) parts.

	$\sim$
1	0
0	1

$\wedge$	1	0
1	1	0
0	0	0

$\vee$	1	0
1	1	1
0	1	0

$\supset$	1	0
1	1	0
0	1	1

$\equiv$	1	0
1	1	0
0	0	1

Truth Tables

Let's see this in action. Question: if p and q are true, is '$p \supset \sim q$' true?

p	q	p	$\supset$	$\sim$	q

Truth Tables

Let's see this in action. Question: if p and q are true, is '$p \supset \sim q$' true?

p	q	p	$\supset$	$\sim$	q
1	1				

Truth Tables

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p	q	p	$\supset$	$\sim$	q
1	1	1			

Truth Tables

Let's see this in action. Question: if p and q are true, is '$p \supset \sim q$' true?

p	q	p	$\supset$	$\sim$	q
1	1	1			1

Truth Tables

Let's see this in action. Question: if p and q are true, is '$p \supset \sim q$' true?

p	q	p	$\supset$	($\sim$	q)
1	1	1		0	1

Truth Tables

Let's see this in action. Question: if p and q are true, is '$p \supset \sim q$' true?

p	q	p	⊃	(∼	q)
1	1	1	0	0	1

So, the answer is: No.

Question: if p is true and q and r are false, is '$p \supset (q \supset r)$' true?

p	q	r	p	⊃	(q	⊃	r)

Truth Tables

Let's see this in action. Question: if p and q are true, is '$p \supset \sim q$' true?

p	q	p	$\supset$	($\sim$	q)
1	1	1	0	0	1

So, the answer is: No.

Question: if p is true and q and r are false, is '$p \supset (q \supset r)$' true?

p	q	r	p	$\supset$	(q	$\supset$	r)
1	0	0					

Truth Tables

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1	0	0	1		0		0

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Let's see this in action. Question: if p and q are true, is '$p \supset \sim q$' true?

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1	0	0	1		0	1	0

Truth Tables

Let's see this in action. Question: if p and q are true, is '$p \supset \sim q$' true?

p	q	p	⊃	(∼	q)
1	1	1	0	0	1

So, the answer is: No.

Question: if p is true and q and r are false, is '$p \supset (q \supset r)$' true?

p	q	r	p	⊃	(q	⊃	r)
1	0	0	1	1	0	1	0

So, the answer is: Yes!

Validity (Truth Tables)

An argument is valid if there is no case of the premises being true and the conclusion false. We can capture this idea with truth tables.

Semantic Consequence (Truth Table)

Let $\Gamma \vDash A$ mean that A is a semantic consequence of Γ, where Γ is a set of formulae of $\mathcal{L}_\rho$. If $\Gamma \vDash A$ is not the case, we write $\Gamma \nvDash A$. $\Gamma \vDash A$ iff there is no line of the relevant truth table in which all the formulae in Γ are true and A false.

Valid Formulae and Contradictions

A is a *validity* iff A is the semantic consequence of the empty set of premises ($\vDash A$) iff, the truth table for A contains no line in which A is false. A is a *contradiction* iff A is false on every line of the relevant truth table.

Testing for Validity (Truth Tables)

Let's test: $\vdash (p \supset q) \supset ((q \supset r) \supset (p \supset r))$

p	q	r	$(p$	$\supset$	$q)$	$\supset$	$((q$	$\supset$	$r)$	$\supset$	$(p$	$\supset$	$r))$
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	0	1	1	1	1	1	0	0	1	1	0	0
1	0	1	1	0	0	1	0	1	1	1	1	1	1
1	0	0	1	0	0	1	0	1	0	0	1	0	0
0	1	1	0	1	1	1	1	1	1	1	0	1	1
0	1	0	0	1	1	1	1	0	0	1	0	1	0
0	0	1	0	1	0	1	0	1	1	1	0	1	1
0	0	0	0	1	0	1	0	1	0	1	0	1	0

Testing for Validity (Truth Tables)

We can also test arguments. Recall the argument from earlier.

(1′) Sally is a cat
(2′) If Sally is a cat, then Sally is a mammal
∴ (3′) Sally is a mammal

Letting $p =$‘Sally is a cat’ and $q =$‘Sally is a mammal’, and formalizing (2′) as ‘$(p \supset q)$’, we then ask $\{p, (p \supset q)\} \vdash q$?

Testing for Validity (Truth Tables)

Question: $\{p, (p \supset q)\} \vdash q$?

p	q	p	,	$(p$	$\supset$	$q)$	q
1	1	1		1	1	1	1
1	0	1		1	0	0	0
0	1	0		0	1	1	1
0	0	0		0	0	0	0

Testing for Validity (Truth Tables)

Question: $\{p, (p \supset q)\} \vdash q$?

p	q	p	,	$(p$	$\supset$	$q)$	q
1	1	1		1	1	1	1
1	0	1		1	0	0	0
0	1	0		0	1	1	1
0	0	0		0	0	0	0

Yes: on every line where the premises are true, the conclusion is true.

Interpretation Semantics

An alternative semantics uses an interpretation function v.

Interpretation

An interpretation v is a function which assigns, for every sentence letter p, either 1 (true) or 0 (false) and which satisfies the following constraints, where we write $v(p) = 1$ if v assigns 1 to p and $v(p) = 0$ if v assigns 0 to p.

($\sim^v$) $v(\sim p) = 1$ iff $v(p) = 0$; and 0 otherwise

($\wedge^v$) $v(p \wedge q) = 1$ iff $v(p) = 1$ and $v(q) = 1$; and 0 otherwise

($\vee^v$) $v(p \vee q) = 1$ iff $v(p) = 1$ or $v(q) = 1$; and 0 otherwise

($\supset^v$) $v(p \supset q) = 1$ iff $v(p) = 0$ or $v(q) = 1$; and 0 otherwise

($\equiv^v$) $v(p \equiv q) = 1$ iff $v(p) = v(q)$; and 0 otherwise

Propositional Logic: Semantics

We can also define some important semantic notions in terms of v.

Semantic Consequence

$\Gamma \vDash A$ iff there is no interpretation v which makes all the formulae in Γ true and A false.

Tautology and Contradiction

A is a validity iff A is the semantic consequence of the empty set of premises ($\vDash A$) iff, for every interpretation v, $v(A) = 1$. A is a contradiction iff, for every interpretation v, $v(A) = 0$.

Propositional Logic: Semantics—Examples

Example I (Good)

Let $\Gamma = \{p, p \supset q\}$. In which case, $\Gamma \vDash q$. Why? Well, $\Gamma \vDash q$ holds iff

there is no v such that $v(p) = 1$, $v(p \supset q) = 1$ and $v(q) = 0$

Suppose $v(p) = 1$, $v(p \supset q) = 1$.
If $v(p \supset q) = 1$, then $v(p) = 0$ or $v(q) = 1$.
Since $v(p) = 1$, if $v(p \supset q) = 1$, then $v(q) = 1$.
So, $\Gamma \vDash q$

Propositional Logic: Semantics—Examples

Example II (Bad)
Let $\Gamma = \{p, q\}$. In which case, $\Gamma \not\models r$. Why? Well, $\Gamma \not\models r$ holds iff

there is a v such that $v(p) = 1$, $v(q) = 1$ and $v(r) = 0$

And there is, trivially! Let v be such that $v(p) = v(q) = 1$ and $v(r) = 0$.

Propositional Logic: The Semantic Definition

We now know the language $\mathcal{L}_\rho$ and we have two semantics.

But what is Propositional *Logic itself*?

Very generally, a logic $\mathcal{L}$ is just a special set of *wff* in the language of $\mathcal{L}$.

Propositional Logic (Semantic Definition)

Propositional Logic is the set of all tautologies (logical truths), i.e., the set of all formulae $\phi \in \mathcal{L}_\rho$ such that, for any v, $v(\phi) = 1$. (Equivalently, the set of all formulae $\phi \in \mathcal{L}_\rho$ such that the truth table for ϕ contains no line in which ϕ is false.)

Propositional Logic: Proof Theory

We can also think about Propositional Logic in a completely different way: we can think in terms of proofs.

With proofs, we are not concerned with what the symbols mean, i.e., not semantics. We are only concerned with how we can manipulate the symbols in $\mathcal{L}_\rho$ purely syntactically.

There are different approaches to proof theory, but in this course we focus on what are called Systems/Axiom Systems/Hilbert Systems.

Propositional Logic: Proof Theory

Each system has an *axiomatic basis*. This comprises of:

(1) Specification of the language (in our case, $\mathcal{L}_\rho$).

(2) A set of *wff*, known as axioms.

(3) A set of transformations rules, or inference rules.

A *wff* is a theorem if it can be obtained by the axioms by applications of the transformation rules. If A is a theorem, we write $\vdash A$.

Propositional Logic: Proof Theory

Each axiom is an individual *wff*. For instance, the following axiom is sometimes given when axiomatizing Propositional Logic:

$$(p \vee p) \supset p$$

Transformation rules tell us what we are allowed to 'conclude' as a theorem, given what we already have shown to be a theorem.

We will use the following one a lot.

Rule of Uniform Substitution (US)

The result of uniformly replacing any variable or variables $p_1, \ldots, p_n$ in a theorem by any *wff* $\beta_1, \ldots, \beta_n$ respectively is itself a theorem.

Summary

1. We looked at what an argument, what makes a good argument, distinguishing valid arguments from formally valid ones.
2. We looked at the language of Propositional Logic, $\mathcal{L}_\rho$.
3. We looked at a semantics for $\mathcal{L}_\rho$ in terms of truth tables.
4. We looked at a semantics for $\mathcal{L}_\rho$ in terms of valuations v.
5. We looked at how to define *valid formula*, *valid argument*, *tautology*, and *contradiction* using truth tables and valuations.
6. We looked at a birds-eye view of proof theory for Propositional Logic, focusing on axiom systems.