Not-so-Simple Quantified Modal Logic

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1. Recap

1.1 Simple Quantified Modal Logic

SQML is the result of supplementing classical quantifier logic with modal operators, governed by some natural principles for modality. SQML + S, where S is any normal modal system can be defined as follows.

(**Syntactically**) Define the a modal system with the axioms and rules of LPC, the axioms for S (i.e., the axioms for K, T, S4, B, or S5), (N) and (BF).

The logic SQML + S is the set of *theorems* of that system.

(**Semantically**) Define the notion of an S-frame $\langle W, D, R\rangle$ and model $\langle W, D, R, v\rangle$.[1] A logical truth for a class of frames are *wff* true in every model based on any frame.

The logic SQML + S is the set of logical truths (defined in terms of S-frames)

To define SQML + S with identity (SQML$^{=}$ + S) we supplement these definitions.

(**Syntactically**) SQML$^{=}$ + S is the set of theorems of SQML + S supplemented with

(I1) $x = x$ is an axiom.

(I2) $x = y \supset (\alpha \supset \beta)$, if α and β differ only in that α has free x in zero or more places where β has free y.

(**Semantically**) Extend the role of v to assigning extensions to $\ulcorner = \urcorner$:

- $v(=)$ is the set of triples $\langle u, u, w\rangle$, for every $u \in D$ and $w \in W$.
- $\mathfrak{M}, w, \mu \vDash x = y$ iff $\langle \mu(x), \mu(y), w\rangle \in v(=)$

[1] In terms of restrictions on R:
K = no restriction
T = reflexive
S4 = reflexive and transitive
B = reflexive and symmetric
S5 = euclidean.

1.2 Necessitism, Contingentism, Actualism and Possibilism

Possibilism: There are *possibilia*, i.e., things that are not actual but could have been.

Actualism: There could not have been *possibilia*.

- Various ways of spelling this out, e.g., existence vs. subsistence, concreteness.

Neccessitism (N): $L\forall xL\exists y(y = x)$

(*To be read*: Necessarily, everything, necessarily is something, i.e., exists.)

Contingentism (C): $M\exists xM\neg\exists y(y = x)$

(*To be read*: Possibly, something, possibly is nothing, i.e., doesn't exist.)

1.3 Necessitism, Contingentism, Actualism and Possibilism and SQML

SQML is problematic for contingentists and actualists.

1. *Converse/Barcan Formula Problem*. The Barcan Formula is valid in SQML:

   (BF) $M\exists x\alpha \supset \exists xM\alpha$

   An instance of this, where $\ulcorner Cx \urcorner$ is $\ulcorner x$ is Wittgenstein's third child$\urcorner$:

   (BF*) $M\exists xCx \supset \exists xMCx$. (*Absurd!*)

   The converse Barcan formula is also valid in SQML:

   (CBF) $\exists xM\alpha \supset M\exists\alpha$

   An instance of this, in SQML$^=$ is the following.

   (CBF*) $\exists xM{\sim}\exists y(y = x) \supset M\exists x{\sim}\exists y(y = x)$ (*Absurd!*)

2. *Simple Argument for Necessitism*. Necessitism is a theorem of SQML$^=$ given:

   (I1) $x = x$ is an axiom

   (N) If $\vdash \alpha$, then $\vdash L\alpha$

   ($\forall$1) $\forall x\alpha \supset \alpha[y/x]$ is an axiom.

2. Free Quantified Modal Logic

We must reformulate SQML if we are contingentists or actualists. The most common approach is to develop a Free Quantified Modal Logic (FQML).[2]

2.1 FQML$^=$ Syntactic

We need to specify the language, axioms and rules.

The language of FQML$^=$ is easy: it is the *same* (lexicon and grammar) as SQML$^=$.

The non-modal axioms of FQML$^=$ differ from SQML$^=$. To specify them, we need to define a so-called 'existence predicate'.

DEFINITION 1. (Existence) Let $\ulcorner \mathrm{E}x \urcorner$ abbreviate $\ulcorner \exists y(y = x) \urcorner$.[3]

With this, we are now in a position to define the axioms of FQML$^=$.

DEFINITION 2. (Axioms of FQML$^=$). The axioms of FQML + S, where S is a normal modal system are all and only the following.

[2] Not the only way. Two other options: Kripke's generality interpretation, see H&C pp. 304–6, and restricted (N), see (Menzel, 2023: §4.3).

[3] Another option: define primitive logical existence predicate, see H&C, p. 292–3. We would need to do this for FQML *without* identity.

(S′) Any LPC substitution-instance of a theorem of S.

(∀1E) $\forall x\alpha \supset (\mathrm{E}y \supset \alpha[y/x])$.

(VQ) $\forall x\alpha \equiv \alpha$, provided x is not free in α.

($\forall^{\supset}$) $\forall x(\alpha \supset \beta) \supset (\forall x\alpha \supset \forall x\beta)$.

(UE) $\forall x \mathrm{E}x$.

(I1) $x = x$ is an axiom.

(I2) $x = y \supset (\alpha \supset \beta)$, if α and β differ only in that α has free x in zero or more places where β has free y.

We also need transformation rules.

DEFINITION 3. (Transformation Rules FQML$^=$) The transformation rules of FQML$^=$ + *S* are the same as SQML$^=$ + *S* with the following addition:

(UG) If $\vdash \alpha$, then $\vdash \forall x\alpha$

(UGL$\forall^n$) $\vdash \alpha_1 \supset L(\alpha_2 \supset ... \supset L(\alpha_n \supset L\beta)...) \to \vdash \alpha_1 \supset L(\alpha_2 \supset ... \supset L(\alpha_n \supset L\forall x\beta)...)$, where x is not free in $\alpha_1, ..., \alpha_n$.

The notion of a theorem: a theorem is an *wff* which follows from applications of the transformation rules (Def. 3) to axioms (Def. 2). We then define the logic.

DEFINITION 4. (FQML$^=$) The logic FQML + *S* is the set of theorems.

2.2 FQML to the rescue?

What was the point of all that?

1. *It blocks the simple argument for necessitism.*

The simple argument starts from:

(1) $\vdash x = x$

This is fine. (1) is true for both FQML$^=$ and SQML$^=$. The argument then proceeds:

(2) $\vdash x = x \supset \exists y(y = x)$

(2) is a theorem of SQML$^=$ because it is the contrapositive of:

(3) $\vdash \forall y{\sim}(y = x) \supset {\sim}x = x$

And (3) is an instance of an axiom of LPC. *There you go!*

Whilst (3) is true for SQML$^=$, it fails to be true for FQML$^=$. Instead, we have:

(4) $\vdash \forall y{\sim}(y = x) \supset (\mathrm{E}x \supset {\sim}x = x)$

From which we can only derive the trivial and metaphysical insigificant:

(5) $\vdash (\mathrm{E}x \wedge x = x) \supset \exists y(y = x)$

2. *The Barcan Formula is not a theorem of FQML$^=$.*

Try as hard as you might, we cannot prove (BF) in FQML$^=$.

To *prove* that you cannot we need a sound semantics for FQML$^=$.[4]

[4] Soundness gets you: if $\vdash \alpha$, then $\vDash \alpha$. Thus, if $\nvDash \alpha$, then $\nvdash \alpha$.

2.3 FQML$^=$ Semantically

We need to specify a different kind of model. The definition of a frame remains:

DEFINITION 5. (A Frame) Let a *frame* $\mathcal{F}$ be a tuple $\langle W, R\rangle$, where W is a non-empty set and R a binary relation on W.

Instead of the usual models, we define a Kripke Model.

DEFINITION 6. (Kripke Model) Let a *Kripke Model* $\mathfrak{M}^{\mathcal{K}}$ be a a tuple $\langle W, R, D, d, v\rangle$, where W and R are as defined above, and where D a non-empty set, d a function from $w \in W$ to subsets of D, D_w, for each w, and v a valuation function such that v assigns, for every n-place predicate ϕ in the language of FQML$^=$, a set of $n+1$ tuples $\langle u_1, ..., u_n, w\rangle$, for each $w \in W$. In particular:

($=^v$) $v(=)$ is the set of triples $\langle u, u, w\rangle$, for every $u \in D$ and $w \in W$.

For a semantics, we need to define truth in a Kripke Model.

DEFINITION 7. (Truth in $\mathfrak{M}^{\mathcal{K}}$) Let μ be an assignment to the variables such that for each variable x, $\mu(x) \in D$. Then, every *wff* has a truth-value at a world $w \in W$, in the model $\mathfrak{M}^{\mathcal{K}}$, under an assignment μ, as determined:

(ϕ^v) $v_\mu(\phi x_1 ... x_n, w) = 1$ if $\langle \mu(x_1), ..., \mu(x_n), w\rangle \in v(\phi)$.

($\sim^v$) $v_\mu(\sim\alpha, w) = 1$ if $v_\mu(\alpha, w) = 0$.

... and so on for the other logical connectives ...

($\forall^v$) $v_\mu(\forall x\alpha, w) = 1$ if $v_\rho(\alpha, w) = 1$, for any x-alternative ρ of μ: $\rho(x) \in D_w$.

($\exists^v$) $v_\mu(\exists x\alpha, w) = 1$ if $v_\rho(\alpha, w) = 1$, for some x-alternative ρ of μ: $\rho(x) \in D_w$.

(L^v) $v_\mu(L\alpha, w) = 1$ if $v_\mu(\alpha, w') = 1$ for every w' such that Rww'.

...0 otherwise.

As usual, we say that α is valid in a model, if α is true at every world, under any assignment. We write: $\mathfrak{M}^{\mathcal{K}} \vDash \alpha$ if so. If α is valid in any model based on a class of frames $\mathcal{F}$, we say it is valid in $\mathcal{F}$. Crucially, we have soundness for systems of FQML$^=$ relative to semantics defined over specific classes of frames.

Soundness: If α is a theorem of FQML$^=$ + *S*, then α is valid in the class of frames associated with *S*, where *S* is either K, T, S4, B, or S5.

Given soundness, if we can find a Kripke Model $\mathfrak{M}^{\mathcal{K}}$ in which (BF) fails, then (BF) is not a theorem of FQML$^=$. Here's a Kripke Model invalidating (BF).

(*) $\mathfrak{M}^{\mathcal{K}} = \langle W, R, D, d, v\rangle$, where $W = \{1, 2\}$, R is universal, $D = \{3, 4\}$, $d : d(1) = \{3\}$ and $d(2) = \{4\}$, and $v(\phi) = \{\langle \varnothing, 1\rangle, \langle 4, 2\rangle\}$. Thus:

- $\mathfrak{M}^{\mathcal{K}}, 1, \mu \vDash M\exists x\phi x$, for any μ; but
- $\mathfrak{M}^{\mathcal{K}}, 1, \mu \nvDash \exists x M\phi x$

3. FQML$^=$: end of the worries? or just more worries?

Not everyone thinks FQML$^=$ is the right choice for the contingentist. Some think that it is the best option *therefore* we should abandon contingentism/actualism:

> The restrictions on instantiation (for $\forall$) and generalisation (for $\exists$) complicate quantificational reasoning, at least in modal contexts, and the intended effect is **a loss of logical power**. Since both **simplicity and strength are virtues in a theory**, judged by normal scientific standards, these restrictions in contingentist logic should give one pause (Williamson, 2013: 43)[5]

[5]This argument presupposes Anti-Exceptionalism about logic—we cover this in the next two weeks!

One may also worry about how predication works in FQML$^=$.

Given the identity and modal axioms in FQML$^=$, the following holds.

(NI) $\vdash L\forall x L(x = x \supset Lx = x)$

Consequently, the following also holds:

(!) $\nvdash L\forall x L(x = x \supset \mathrm{E}x)$

Why? If (NI) but $\vdash L\forall x L(x = x \supset \mathrm{E}x)$, then $\vdash L\forall x L\mathrm{E}x$. An alternative explanation: we can easily construct a $\mathfrak{M}^{\mathcal{K}}$ in which $L\forall x L(x = x \supset \mathrm{E}x)$ fails to be valid.

Thus, FQML$^=$ as we have set it up violates **Serious Actualism**.

- But is this a problem? This depends on whether you think Serious Actualism should be logically true, not *just true*.

4. Questions

1. Construct a Kripke Model in which $L\forall x L(x = x \supset \mathrm{E}x)$ is not valid.
2. Is the model given in answer to (1.) of any significance to the debate?

References

Menzel, Christopher (2023). The Possibilism-Actualism Debate. In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta and Uri Nodelman. Spring 2023. Metaphysics Research Lab, Stanford University.

Williamson, Timothy (2013). *Modal Logic as Metaphysics*. Oxford University Press.